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G. N. Bycroft

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FORCED VIBRATIONS OF A RIGID CIRCULAR PLATE ON A SEMI-INFINITE ELASTIC SPACE AND ON AN ELASTIC STRATUM

By G. N. BYCROFT

Dominion Physical Laboratory, Department of Scientific and Industrial Research, Lower Hutt, New Zealand

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CONTENTS

The impedance of a rigid circular plate attached to the free surface of a semi-infinite elastic space or an elastic stratum is determined for its four degrees of freedom. The solution of the dual integral equations arising from this mixed boundary-value problem is avoided by reference to Rayleigh's reciprocal theorem. This enables the functions of frequency, which determine the in-phase and out-of-phase components of displacement of the plate, to be located between two close bounds and lying much closer to one than to the other. These bounds appear as infinite integrals involving branch functions and are reduced to tractable finite integrals by integration in the complex plane. Dissipation of waves to infinity produces an effective damping, and the added effect of the inclusion of true damping in the medium is discussed.

It is to be expected, of course, that the unloaded rigid plate attached to the free surface of a semi-infinite elastic space does not resonate. The change of impedance of the plate with frequency is found to be similar for the two translations and also similar for the two rotations. Resonance occurs in the case of vertical and horizontal translation of the plate attached to the surface of an elastic stratum. However, it does not exist for rotations of the plate on the stratum. Instead, a maximum in the response appears, this maximum being more defined the greater the ratio of plate diameter to stratum depth. The addition of small true damping in the medium changes the characteristics very little.

Experimental work substantiating these theoretical results, together with a general discussion of the results and their applications in geophysics and engineering, is being published shortly.

1. Introduction

Analytical approaches follow from the fundamental work of Lamb (1904) on the propagation of elastic waves. Of the eight cases treated in this paper, two have been considered before. Reissner (1936) considered the case of the harmonically forced vertical translation of a rigid circular plate attached to an elastic half-space but used greatly simplified boundary conditions and his results are of qualitative interest only. In a further paper Reissner (1937) considered the torsional oscillations of an elastic half-space but again simplified the boundary conditions. However, Reissner & Sagoci (1944) considered this latter problem again, using a system of oblate spheroidal co-ordinates and producing a full solution. Unfortunately, this method is inapplicable to the other cases. Marguerre (1933) treats wave propagation in elastic strata but not the special sources required in this work.

Although circular bases only have been treated the results will be at least qualitatively true for other base shapes.

The analysis is mainly concerned with the determination of two functions of frequency called f_1 and f_2 being effectively the in-phase and out-of-phase components of displacement of the unloaded plate. The effect of loading the plate by the addition of mass or moment of inertia follows simply from these two functions. The adoption of certain dimensionless variables decreases the number of parameters and the following notation is convenient.

Notation

(1) $\lambda, \mu =$ Lamé's elastic constants of the medium.

(2) $\rho =$ density of the medium.

(3) $p =$ angular velocity of impressed force or couple on the plate.

(4) $\dfrac{\rho p^2}{(\lambda+2\mu)} = h^2, \quad \dfrac{\rho p^2}{\mu} = k^2, \quad \dfrac{h}{k} = \sqrt{\left(\dfrac{\mu}{\lambda+2\mu}\right)} = \tau.$

(5) $r_0 =$ radius of plate.

(6) $\delta =$ depth of stratum.

(7) $\delta/r_0 = R$.

(8) u, v, w are displacements of a point r, θ, z in these co-ordinate directions.

(9) U, V, W are displacements of the centre of the plate in the respective directions.

(10) $\phi =$ angle of rotation of the plate.

(11) $P =$ amplitude of the exciting force.

(12) $M =$ amplitude of the exciting couple.

(13) $kr_0 = a_0, \quad kr = a, \quad k\delta = \gamma.$

(14) f_1, f_2 are functions of a_0 and τ and are such that the translation or rotation of the unloaded plate may be given by the real part of

(15) U, V or $W = \dfrac{P\,\mathrm{e}^{\mathrm{i}pt}}{\mu r_0}(f_1+\mathrm{i}f_2)$ or $\phi = \dfrac{M\,\mathrm{e}^{\mathrm{i}pt}}{\mu r_0^3}(f_1+\mathrm{i}f_2)$, where f_1 and f_2 are different for each mode; the real part is implied in the following analysis.

(16) $m_0 =$ mass attached to the plate.

(17) $I_0 =$ moment of inertia of mass.

(18) $A =$ amplitude of vibration of plate.

(19) $L =$ average power input.

(20) $\psi =$ phase angle between exciting force and the resulting displacement.

(21) $b = \dfrac{m_0}{\rho r_0^3}, \quad b' = \dfrac{I_0}{\rho r_0^5}.$

It is easy to show that the following relations hold for the cases of translation when the plate is loaded with a mass:

(22) $$\tan\psi = \frac{f_2}{f_1+ba_0^2(f_1^2+f_2^2)},$$

(23) $$A = \frac{P}{\mu r_0}\sqrt{\left\{\frac{f_1^2+f_2^2}{(1+ba_0^2f_1)^2+(ba_0^2f_2)^2}\right\}},$$

(24) $$L = \frac{P^2}{2r_0^2\sqrt{(\rho\mu)}}\left\{\frac{a_0 f_2}{(1+ba_0^2f_1)^2+(ba_0^2f_2)^2}\right\}.$$

When considering rotation replace $\dfrac{P}{\mu r_0}$ by $\dfrac{M}{\mu r_0^3}$ and b by b'.

2. Solutions of the elastic wave equations

The equations of motion of a linear elastic medium, in cylindrical co-ordinates, may be expressed by the following equations. The notation is that of Love:

$$\rho\frac{\partial^2 u}{\partial t^2}=(\lambda+2\mu)\frac{\partial\Delta}{\partial r}-\frac{2\mu}{r}\frac{\partial\varpi_z}{\partial\theta}+2\mu\frac{\partial\varpi_\theta}{\partial z}, \tag{1}$$

$$\rho\frac{\partial^2 v}{\partial t^2}=(\lambda+2\mu)\frac{1}{r}\frac{\partial\Delta}{\partial\theta}-2\mu\frac{\partial\varpi_r}{\partial z}+2\mu\frac{\partial\varpi_z}{\partial r}, \tag{2}$$

$$\rho\frac{\partial^2 w}{\partial t^2}=(\lambda+2\mu)\frac{\partial\Delta}{\partial z}-\frac{2\mu}{r}\frac{\partial}{\partial r}(r\varpi_\theta)+\frac{2\mu}{r}\frac{\partial\varpi_r}{\partial\theta}, \tag{3}$$

$$\Delta=\frac{1}{r}\frac{\partial(ru)}{\partial r}+\frac{1}{r}\frac{\partial v}{\partial\theta}+\frac{\partial w}{\partial z}, \tag{4}$$

$$2\varpi_\theta=\frac{\partial u}{\partial z}-\frac{\partial w}{\partial r},\quad 2\varpi_r=\frac{1}{r}\frac{\partial w}{\partial\theta}-\frac{\partial v}{\partial z}, \tag{5}$$

$$2\varpi_z=\frac{1}{r}\left\{\frac{\partial(rv)}{\partial r}-\frac{\partial u}{\partial\theta}\right\}. \tag{6}$$

It has been shown by Sezawa (1929) that u, v, w are particular solutions of the above equations, where

$$u=u_1+u_2+u_3,\quad v=v_1+v_2+v_3,\quad w=w_1+w_2+w_3, \tag{7}$$

$$u_1=-A_m\frac{1}{h^2}\frac{\partial}{\partial r}H_m^{(2)}(xr)\,e^{-\alpha z+ipt}\,{\cos\atop\sin}\,m\theta, \tag{8}$$

$$v_1=A_m\frac{m}{h^2}\frac{H_m^{(2)}(xr)}{r}\,e^{-\alpha z+ipt}\,{\sin\atop -\cos}\,m\theta, \tag{9}$$

$$w_1=A_m\frac{\alpha}{h^2}H_m^{(2)}(xr)\,e^{-\alpha z+ipt}\,{\cos\atop\sin}\,m\theta, \tag{10}$$

$$u_2=B_m\frac{m}{x^2}\frac{H_m^{(2)}(xr)}{r}\,e^{-\beta z+ipt}\,{\cos\atop\sin}\,m\theta, \tag{11}$$

$$v_2=-B_m\frac{1}{x^2}\frac{\partial}{\partial r}H_m^{(2)}(xr)\,e^{-\beta z+ipt}\,{\sin\atop -\cos}\,m\theta, \tag{12}$$

$$w_2=0, \tag{13}$$

$$u_3=C_m\frac{\beta}{mk^2}\frac{\partial}{\partial r}H_m^{(2)}(xr)\,e^{-\beta z+ipt}\,{\cos\atop\sin}\,m\theta, \tag{14}$$

$$v_3=-C_m\frac{\beta}{k^2}\frac{H_m^{(2)}(xr)}{r}\,e^{-\beta z+ipt}\,{\sin\atop -\cos}\,m\theta, \tag{15}$$

$$w_3=-C_m\frac{x^2}{mk^2}H_m^{(2)}(xr)\,e^{-\beta z+ipt}\,{\cos\atop\sin}\,m\theta, \tag{16}$$

$$\alpha=+(x^2-h^2)^{\frac{1}{2}},\quad \beta=+(x^2-k^2)^{\frac{1}{2}}. \tag{17}$$

m, x are arbitrary parameters, A_m, B_m, C_m are arbitrary constants with respect to r, z, θ, t, and may be taken as functions of x,

$$H_m^{(2)}(xr)=J_m(xr)-iY_m(xr), \tag{18}$$

i.e. a second Hankel function.

It will be shown that combinations of these three solutions may be made to fit the various boundary conditions considered in this paper. In general, the method will be to integrate the above solutions with respect to the arbitrary parameter x, thus obtaining a more generalized solution, and, by using the Fourier–Bessel integral theorem, choose the functions $A_m(x)$, $B_m(x)$, $C_m(x)$ to satisfy the various boundary conditions. The resulting infinite integral, however, must be further examined in order to determine its significant value. This is the method adopted by Lamb (1904). It is straightforward, but perhaps not as elegant as a Hankel transform approach would be. This latter method could certainly be used for rotationally symmetric cases, i.e. $m = 0$, as it is possible in these cases to express the displacement as derivatives of two potentials Ψ and X which satisfy

$$\left.\begin{aligned}(\nabla^2+h^2)\,\Psi = 0, \quad (\nabla^2+k^2)\,X = 0,\\ \nabla^2 = \frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}.\end{aligned}\right\} \tag{19}$$

When $m \neq 0$ it is not clear whether a Hankel transform of the equations would provide a readily available result.

The notation relating to stresses is that used by Love, and the following well-known relations between stress and displacement hold:

$$\left.\begin{aligned}\widehat{rr} &= \lambda\Delta+2\mu\frac{\partial u}{\partial r},\\ \widehat{\theta\theta} &= \lambda\Delta+2\mu\left(\frac{1}{r}\frac{\partial v}{\partial \theta}+\frac{u}{r}\right),\\ \widehat{zz} &= \lambda\Delta+2\mu\frac{\partial w}{\partial z},\\ \widehat{\theta z} &= \mu\left(\frac{1}{r}\frac{\partial w}{\partial \theta}+\frac{\partial v}{\partial z}\right),\\ \widehat{zr} &= \mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\right),\\ \widehat{r\theta} &= \mu\left(\frac{\partial v}{\partial r}-\frac{v}{r}+\frac{1}{r}\frac{\partial u}{\partial \theta}\right),\\ \Delta &= \frac{\partial u}{\partial r}+\frac{u}{r}+\frac{1}{r}\frac{\partial v}{\partial \theta}+\frac{\partial w}{\partial z}.\end{aligned}\right\} \tag{20}$$

The values of $A_m(x)$, $B_m(x)$, $C_m(x)$, necessary to satisfy the boundary conditions of the various cases, are considered under four separate headings corresponding to the four degrees of freedom of the plate. Some general remarks may be made here regarding the three solutions. Only the $J_m(xr)$ part of the Hankel function is needed. For axial symmetry $m = 0$ and for rotation about a horizontal axis or translation in a horizontal plane $m = 1$.

For motion of the plate on a semi-infinite space the solutions containing the factors, $\exp(-\alpha z)$, $\exp(-\beta z)$, indicating no boundary at a great depth, are used. However, for motion of the plate on an elastic stratum, it is necessary to compound these three solutions with the corresponding ones involving $\exp(\alpha z)$, $\exp(\beta z)$.

The following results are needed often in the theory.

(*a*) *Dual integral equations*

Busbridge (1938) has shown that the solution of the pair of integral equations

$$\int_0^\infty y^\alpha f(y)\, J_\nu(sy)\, \mathrm{d}y = g(s) \quad (0 \leqslant s \leqslant 1), \tag{21}$$

$$\int_0^\infty f(y)\, J_\nu(sy)\, \mathrm{d}y = 0 \quad (s > 1), \tag{22}$$

where $g(s)$ is given and $f(y)$ is to be found, is given by

$$f(x) = \frac{2^{-\frac{1}{2}\alpha} x^{-\alpha}}{\Gamma(1+\frac{1}{2}\alpha)} \Big[x^{1+\frac{1}{2}\alpha} J_{\nu+\frac{1}{2}\alpha}(x) \int_0^1 y^{\nu+1}(1-y^2)^{\frac{1}{2}\alpha} g(y)\, \mathrm{d}y$$
$$+ \int_0^1 u^{\nu+1}(1-u^2)^{\frac{1}{2}\alpha}\, \mathrm{d}u \int_0^1 g(yu)\, (xy)^{2+\frac{1}{2}\alpha} J_{\nu+1+\frac{1}{2}\alpha}(xy)\, \mathrm{d}y \Big], \tag{23}$$

valid for $\alpha > -2$ or $(-\nu-1) < (\alpha - \frac{1}{2}) < (\nu+1)$. The function $g(y)$ must be integrable over the interval $(0, 1)$.

(*b*) *Sonine's first finite integral*

$$J_{\mu+\nu+1}(z) = \frac{z^{\nu+1}}{2^\nu \Gamma(\nu+1)} \int_0^{\frac{1}{2}\pi} J_\mu(z \sin\theta) \sin^{\mu+1}\theta \cos^{2\nu+1}\theta\, \mathrm{d}\theta, \tag{24}$$

where both $\mathscr{R}(\mu)$ and $\mathscr{R}(\nu)$ exceed -1.

(*c*) *Hankel functions*

The following properties of Hankel functions are used in the analysis, the notation being that of Watson (1944):

$$\left.\begin{aligned} H_n^{(1)}(x) &= J_n(x) + \mathrm{i}Y_n(x), \\ H_n^{(2)}(x) &= J_n(x) - \mathrm{i}Y_n(x), \end{aligned}\right\} \tag{25}$$

and when x becomes large

$$\left.\begin{aligned} H_n^{(1)}(x) &\sim \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \mathrm{e}^{\mathrm{i}(x-\frac{1}{2}n\pi-\frac{1}{4}\pi)}, \\ H_n^{(2)}(x) &\sim \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \mathrm{e}^{-\mathrm{i}(x-\frac{1}{2}n\pi-\frac{1}{4}\pi)}. \end{aligned}\right\} \tag{26}$$

The factor $\mathrm{e}^{\mathrm{i}pt} H_n^{(1)}(x)$ represents a wave progressing towards the origin and $\mathrm{e}^{\mathrm{i}pt} H_n^{(2)}(x)$ a wave progressing from the origin:

$$\left.\begin{aligned} H_0^{(1)}(-x) &= -H_0^{(2)}(x), \\ H_1^{(1)}(-x) &= H_1^{(2)}(x). \end{aligned}\right\} \tag{27}$$

(*d*) *Fourier–Bessel integral expansion*

The following expansion is true if $n \geqslant -\frac{1}{2}$:

$$f(x) = \int_0^\infty J_n(tx)\, t \left\{ \int_0^\infty f(x')\, J_n(tx')\, x'\, \mathrm{d}x' \right\} \mathrm{d}t. \tag{28}$$

(*e*) *Contour integration with branch-points*

Infinite integrals involving the branch points $\alpha = +(x^2 - h^2)^{\frac{1}{2}}$, $\beta = +(x^2 - k)^{\frac{1}{2}}$, occur repeatedly in the following analysis. It is desirable to integrate these integrals around an infinite semicircle in the upper half-plane and along the horizontal axis (figure 1). By staying on one branch of the above branch functions it follows that integrands containing these functions may be integrated around this contour if the sign of the radical is changed to the left of the branch-points $-h$ or $-k$.

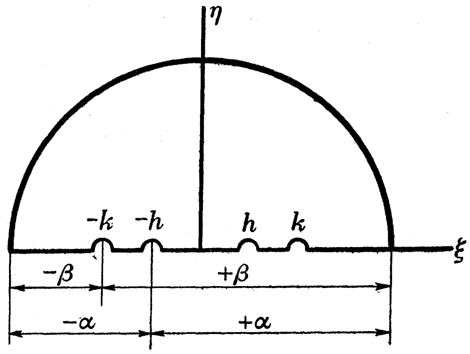

FIGURE 1. The signs to be given to the radicals α and β.

3. VERTICAL TRANSLATION

Semi-infinite elastic space

When the light rigid circular plate is attached to the free surface of an elastic half-space and excited by a symmetrical vertical force a solution may be obtained from the particular solutions given by equations (8 to 10) and (14 to 16).

The motion is rotationally symmetric, i.e. $m = 0$, $v = 0$, and the other displacements follow as

$$u = \left[\frac{Ax\,\mathrm{e}^{-\alpha z}}{h^2} - \frac{C\beta x\,\mathrm{e}^{-\beta z}}{k^2}\right] J_1(xr)\,\mathrm{e}^{\mathrm{i}pt}, \tag{29}$$

$$w = \left[\frac{A\alpha\,\mathrm{e}^{-\alpha z}}{h^2} - \frac{Cx^2\,\mathrm{e}^{-\beta z}}{k^2}\right] J_0(xr)\,\mathrm{e}^{\mathrm{i}pt}. \tag{30}$$

(a) *Static displacement of plate*

First, the static displacement of the plate is considered. This, of course, is the well-known Boussinesq problem and is solved here using the dual integral equations already mentioned. If the two arbitrary functions $A(x)$ and $C(x)$ are to be retained the static solutions must be derived from the dynamic ones as follows. Let the frequency tend to zero, i.e. h, k tend to zero, and expand to the first order in h, k:

$$u \sim \left[\frac{Ax}{h^2}\left(1+\frac{h^2 z}{2x}\right) - \frac{Cx^2}{k^2}\left(1-\frac{k^2}{2x^2}+\frac{k^2 z}{2x}\right)\right] \mathrm{e}^{-xz} J_1(xr), \tag{31}$$

$$w \sim \left[\frac{Ax}{h^2}\left(1-\frac{h^2}{2x^2}+\frac{h^2 z}{2x}\right) - \frac{Cx^2}{k^2}\left(1+\frac{k^2 z}{2x}\right)\right] \mathrm{e}^{-xz} J_0(xr). \tag{32}$$

If these are rearranged and
$$\frac{Ax}{h^2} + C\left(\frac{k^2-2x^2}{2k^2}\right) = A_1, \tag{33}$$

$$\frac{A-Cx}{2} = B_1, \quad \frac{h^2}{k^2} = \tau^2, \tag{34}$$

then
$$u = (A_1 + B_1 z)\,\mathrm{e}^{-xz} J_1(xr), \tag{35}$$

$$w = \left[A_1 - \frac{B_1(\tau^2+1)}{x(\tau^2-1)} + B_1 z\right] \mathrm{e}^{-xz} J_0(xr). \tag{36}$$

From these displacements and the relations quoted between displacement and stress it follows that the normal and tangential stresses on the free surface, $z = 0$, are given by

$$\widehat{zr} = 2\mu\left[\frac{B_1\tau^2}{(\tau^2-1)} - A_1 x\right] e^{-xz} J_1(xr), \tag{37}$$

$$\widehat{zz} = 2\mu\left[\frac{B_1}{(\tau^2-1)} - A_1 x\right] e^{-xz} J_0(xr). \tag{38}$$

Using this method it is necessary to assume that the shear stress everywhere on the free surface vanishes. This means that horizontal movement can exist under the plate. However, it is indicated later that prevention of this movement by friction will cause little change in the vertical deflexion. Put $\widehat{zr} = 0$ and $A_2(x) = A_1(x)/\tau^2$ and then

$$w = A_2(x) J_0(xr), \tag{39}$$

$$\widehat{zz} = 2\mu A_2(x)\, x(\tau^2-1) J_0(xr). \tag{40}$$

These solutions are generalized by integrating from 0 to ∞ with respect to the arbitrary parameter x and then, if d is the vertical displacement of the plate and r_0 is its radius, $A(x)$ is to be chosen so as to satisfy the remaining two boundary conditions given by

$$w = \int_0^\infty A(x) J_0(xr)\,dx = d \quad (r \leqslant r_0), \tag{41}$$

$$\widehat{zz} = 2\mu(\tau^2-1)\int_0^\infty A(x)\, xJ_0(xr)\,dx = 0 \quad (r > r_0). \tag{42}$$

Put
$$\frac{r}{r_0} = s, \quad xr_0 = y, \tag{43}$$

and the equations become a particular case of the dual integral equations (21) and (22), already quoted when $\alpha = -1$, $\mu = 0$, $g(s) = dr_0$, i.e.

$$\int_0^\infty F(y)\, J_0(sy)\,dy = dr_0 \quad (s \leqslant 1), \tag{44}$$

$$\int_0^\infty F(y)\, yJ_0(sy)\,dy = 0 \quad (s > 1). \tag{45}$$

The solution of these two equations is

$$F(y) = \frac{2dr_0}{\pi}\frac{\sin y}{y}. \tag{46}$$

The normal stress under the plate is now given by

$$\widehat{zz} = \frac{4\mu(\tau^2-1)\,dr_0}{\pi r_0^2}\int_0^\infty \sin y J_0(sy)\,dy$$

$$= \frac{4\mu(\tau^2-1)\,d}{\pi r_0(1-s^2)^{\frac{1}{2}}} \quad \text{(Watson 1944, p. 405).} \tag{47}$$

This equation shows that at the edge of the plate the stress is unbounded.

If the normal stress is integrated over the circle $r \leqslant r_0$, the vertical force P necessary to depress the plate d is found to be

$$P = 8\mu d(\tau^2 - 1) r_0 = \frac{2r_0 dE}{(1-\nu^2)}, \tag{48}$$

the well-known Boussinesq deflexion.

The horizontal movement under the plate is given by

$$u = \frac{2d\tau^2}{\pi}\int_0^\infty \frac{\sin y J_1(sy)\,\mathrm{d}y}{y} = \frac{2d\tau^2 s}{\pi[1+(1-s^2)^{\frac{1}{2}}]} \quad \text{(Watson, p. 405).} \tag{49}$$

At the centre, $s = 0$, the movement is zero increasing to $2d\tau^2/\pi$ at the periphery $s = 1$. Incompressibility of the medium, denoted by Poisson's ratio $\nu = \frac{1}{2}$ or $\tau^2 = 0$, means no horizontal movement under the frictionless plate.

Physically, the stretching of the surface due to the depression under the plate is exactly balanced by the increase in area caused by the Poisson ratio effect. The maximum horizontal movement occurs when $\nu = 0$, i.e. $\tau^2 = \frac{1}{2}$ and $s = 1$ and then $u = d/\pi$. If this horizontal movement was restrained it is unlikely that its effect on the vertical motion would be greater than $d/4\pi^2 \doteqdot d/40$. This effect is neglected in what follows.

(*b*) *Dynamic displacement of plate*

From equations (20), (29) and (30), the relevant dynamic displacements and stresses are

$$u = \left[\frac{A(x)\,x\,\mathrm{e}^{-\alpha z}}{h^2} - \frac{C(x)\,\beta x\,\mathrm{e}^{-\beta z}}{k^2}\right] J_1(xr)\,\mathrm{e}^{\mathrm{i}pt}, \tag{50}$$

$$w = \left[\frac{A(x)\,\alpha\,\mathrm{e}^{-\alpha z}}{h^2} - \frac{C(x)\,x^2\,\mathrm{e}^{-\beta z}}{k^2}\right] J_0(xr)\,\mathrm{e}^{\mathrm{i}pt}, \tag{51}$$

$$\widehat{zz} = \mu\left[\frac{A(x)\,(k^2-2x^2)\,\mathrm{e}^{-\alpha z}}{h^2} + \frac{C(x)\,2\beta x^2\,\mathrm{e}^{-\beta z}}{k^2}\right] J_0(xr)\,\mathrm{e}^{\mathrm{i}pt}, \tag{52}$$

$$\widehat{zr} = \mu\left[-\frac{A(x)\,2x\alpha\,\mathrm{e}^{-\alpha z}}{h^2} - \frac{C(x)\,x(k^2-2x^2)\,\mathrm{e}^{-\beta z}}{k^2}\right] J_1(xr)\,\mathrm{e}^{\mathrm{i}pt}. \tag{53}$$

(i) *Free vibrations. Rayleigh waves.* These are found by equating the surface stresses $\widehat{zz} = \widehat{zr} = 0$ when $z = 0$, yielding the following well-known frequency equation determined by Rayleigh:

$$f(x) = (x^2 - \tfrac{1}{2}k^2)^2 - x^2(x^2-h^2)^{\frac{1}{2}}(x^2-k^2)^{\frac{1}{2}} = 0. \tag{54}$$

This equation has the following real roots

$$\left.\begin{aligned} \nu &= \tfrac{1}{2}, \ \tau^2 = 0, \quad x_1 = 1{\cdot}04678k, \\ \nu &= \tfrac{1}{4}, \ \tau^2 = \tfrac{1}{3}, \quad x_1 = \tfrac{1}{2}(3+\surd 3)^{\frac{1}{2}}k, \\ \nu &= 0, \ \tau^2 = \tfrac{1}{2}, \quad x_1 = \tfrac{1}{2}(3+\surd 5)^{\frac{1}{2}}k. \end{aligned}\right\} \tag{55}$$

If

$$\alpha_1^2 = x_1^2 - h^2, \quad \beta_1^2 = x_1^2 - k^2, \tag{56}$$

the free waves are given by

$$u_0 = D\,\mathrm{e}^{\mathrm{i}pt}\,[-x_1(x_1^2-\tfrac{1}{2}k^2)\,\mathrm{e}^{-\alpha_1 z}+\alpha_1\beta_1 x_1\,\mathrm{e}^{-\beta_1 z}]\,J_1(x_1 r), \tag{57}$$

$$w_0 = D\,\mathrm{e}^{\mathrm{i}pt}\,[-\alpha_1(x_1^2-\tfrac{1}{2}k^2)\,\mathrm{e}^{-\alpha_1 z}+\alpha_1 x_1^2\,\mathrm{e}^{-\beta_1 z}]\,J_0(x_1 r), \tag{58}$$

where D is an arbitrary constant. When $z = 0$ these free wave displacements are

$$u_0 = -D_1 x_1(2x_1^2-k^2-2\alpha_1\beta_1)\,J_1(x_1 r)\,\mathrm{e}^{\mathrm{i}pt}, \tag{59}$$

$$w_0 = D_1 k^2\alpha_1 J_0(x_1 r)\,\mathrm{e}^{\mathrm{i}pt}. \tag{60}$$

(ii) *Forced vibrations*. Generalize the solutions (50) to (53) by integrating with respect to x from 0 to ∞ and

$$u = \mathrm{e}^{\mathrm{i}pt}\int_0^\infty \left[\frac{A(x)\,x\,\mathrm{e}^{-\alpha z}}{h^2}-\frac{C(x)\,\beta x\,\mathrm{e}^{-\beta z}}{k^2}\right]J_1(xr)\,\mathrm{d}x, \tag{61}$$

$$w = \mathrm{e}^{\mathrm{i}pt}\int_0^\infty \left[\frac{A(x)\,\alpha\,\mathrm{e}^{-\alpha z}}{h^2}-\frac{C(x)\,x^2\,\mathrm{e}^{-\beta z}}{k^2}\right]J_0(xr)\,\mathrm{d}x, \tag{62}$$

$$\widehat{zz} = \mu\,\mathrm{e}^{\mathrm{i}pt}\int_0^\infty \left[\frac{A(x)\,(k^2-2x^2)\,\mathrm{e}^{-\alpha z}}{h^2}+\frac{C(x)\,2\beta x^2\,\mathrm{e}^{-\beta z}}{k^2}\right]J_0(xr)\,\mathrm{d}x, \tag{63}$$

$$\widehat{zr} = \mu\,\mathrm{e}^{\mathrm{i}pt}\int_0^\infty \left[\frac{-A(x)\,2\alpha x\,\mathrm{e}^{-\alpha z}}{h^2}-\frac{C(x)\,x(k^2-2x^2)\,\mathrm{e}^{-\beta z}}{k^2}\right]J_1(xr)\,\mathrm{d}x. \tag{64}$$

Stresses and displacements elsewhere than the free surface, $z = 0$, are not required in this problem, and in what follows only these latter displacements are considered. If the other displacements are required they follow from the values of $A(x)$ and $C(x)$, determined later, together with equations (61) to (64).

As already mentioned in the static case, it greatly simplifies the results if it is assumed that $\widehat{zr} = 0$, $z = 0$, i.e. the plate is frictionless, allowing horizontal sliding to take place underneath it. Put $\widehat{zr} = 0$, $z = 0$ and

$$C(x) = \frac{-2\alpha k^2 A(x)}{h^2(k^2-2x^2)} \tag{65}$$

and then

$$u(r,0) = \mathrm{e}^{\mathrm{i}pt}\int_0^\infty \frac{A(x)\,x(k^2-2x^2+2\alpha\beta)\,J_1(xr)\,\mathrm{d}x}{h^2(k^2-2x^2)}, \tag{66}$$

$$w(r,0) = \mathrm{e}^{\mathrm{i}pt}\int_0^\infty \frac{A(x)\,\alpha k^2 J_0(xr)\,\mathrm{d}x}{h^2(k^2-2x^2)}, \tag{67}$$

$$\widehat{zz}(r,0) = \mu\,\mathrm{e}^{\mathrm{i}pt}\int_0^\infty \frac{4A(x)\,[(x^2-\tfrac{1}{2}k^2)^2-\alpha\beta x^2]\,J_0(xr)\,\mathrm{d}x}{h^2(k^2-2x^2)}. \tag{68}$$

It is required that $w(r,0) = d$, $r \leqslant r_0$ and $\widehat{zz} = 0$, $r > r_0$, but these two dual integral equations determining $A(x)$ do not have an easy solution. It may be shown that the solution can be approximated to as closely as desired but the computation necessary is prohibitive. Reissner evaded these mixed boundaries by assuming a constant normal stress over $0 \leqslant r \leqslant r_0$ and taking the displacement of the rigid plate as the displacement at the centre of the circle. This is a very crude approximation to a rigid plate and makes the value of the functions much too high. A very close approximation can be made in the following fashion. Assume that the normal stress is that of the static case which can be solved exactly. Then, at low frequencies, with this stress distribution, the displacement $w(r,0)$ $(r \leqslant r_0)$, will be substantially constant as required.

As the frequency increases $w(r, 0)$ $(r \leqslant r_0)$, will become a function of r. Physically, $r \leqslant r_0$ now bridges a greater part of the wavelength of the waves being propagated outwards. Under a rigid plate the pressure distribution changes with frequency. It will be shown later that by taking a particular average of $w(r, 0)$ over $r \leqslant r_0$ it is possible to specify the deflexion of a rigid plate between two close bounds. For the minute, the problem of the displacements when the stress $\widehat{zz}$ over $r \leqslant r_0$ is that of the static case, is considered, i.e. put

$$\widehat{zz} = \frac{P\,\mathrm{e}^{\mathrm{i}pt}}{2\pi r_0 (r_0^2 - r^2)^{\frac{1}{2}}} \quad (r \leqslant r_0),$$
$$= 0 \quad (r > r_0). \tag{69}$$

Express $\widehat{zz}$ by the Fourier–Bessel integral (equation (28))

$$\widehat{zz} = \int_0^\infty J_0(xr)\, x \left\{ \int_0^{r_0} \frac{P\,\mathrm{e}^{\mathrm{i}pt} J_0(xy)\, y\, \mathrm{d}y}{2\pi r_0 (r_0^2 - y^2)^{\frac{1}{2}}} \right\} \mathrm{d}x. \tag{70}$$

The integral inside the brackets may be evaluated by putting $y = r_0 \sin\phi$ and noticing that it is then the special case of Sonine's first finite integral when $\mu = 0$, $z = xr_0$, $\nu = \frac{1}{2}$.

Then

$$\int_0^{r_0} \frac{J_0(xy)\, y\, \mathrm{d}y}{(r_0^2 - y^2)^{\frac{1}{2}}} = \frac{\sin(xr_0)}{x}. \tag{71}$$

Comparison of equation (68) with equation (70) will show that all the conditions will be satisfied if

$$A(x) = \frac{P \sin(xr_0)\, h^2 (k^2 - 2x^2)}{8\mu\pi r_0 f(x)}, \tag{72}$$

where

$$f(x) = (x^2 - \tfrac{1}{2}k^2)^2 - x^2 \alpha\beta,$$

and then

$$u(r, 0) = \frac{-P\,\mathrm{e}^{\mathrm{i}pt}}{8\mu\pi r_0} \int_0^\infty \frac{x(2x^2 - k^2 - 2\alpha\beta) \sin(xr_0)\, J_1(xr)\, \mathrm{d}x}{f(x)}, \tag{73}$$

$$w(r, 0) = \frac{P\,\mathrm{e}^{\mathrm{i}pt}}{8\mu\pi r_0} \int_0^\infty \frac{\alpha k^2 \sin(xr_0)\, J_0(xr)\, \mathrm{d}x}{f(x)}. \tag{74}$$

Following Lamb, we investigate these integrals in order to determine their relevant value. This uncertainty arises from the fact that $f(x)$ has a real root in the range of integration, this root corresponding to the free waves already discussed. All the boundary conditions are satisfied except the radial boundary at infinity. There are to be no reflexions from infinity, and the above integrals are now exhibited in a form in which the nature of the waves at a large radius may be examined.

Put

$$J_m(xr) = \tfrac{1}{2}[H_m^{(1)}(xr) + H_m^{(2)}(xr)],$$

and equations (73) and (74) become

$$u(r, 0) = \frac{-P\,\mathrm{e}^{\mathrm{i}pt}}{16\mu\pi r_0} \int_{-\infty}^\infty \frac{x(2x^2 - k^2 - 2\alpha\beta) \sin(xr_0)\, H_1^{(1)}(xr)\, \mathrm{d}x}{f(x)}, \tag{75}$$

$$w(r, 0) = \frac{P\,\mathrm{e}^{\mathrm{i}pt}}{16\mu\pi r_0} \int_{-\infty}^\infty \frac{\alpha k^2 \sin(xr_0)\, H_0^{(1)}(xr)\, \mathrm{d}x}{f(x)}. \tag{76}$$

Integrate these around a semi-infinite circle in the upper half-plane, changing the signs of α and β to make the integrand analytic as indicated previously. It can be easily shown that if $r \geqslant r_0$ then the integral vanishes around the infinite semicircle. Now $h < k < x_1$ always, and if the signs of α_1 and β_1 at the pole $x = -x_1$ are changed, the sum of the two residues at $\pm x_1$ for $w(r, 0)$ follows as

$$\frac{\alpha_1 k^2 \sin(x_1 r_0)\, H_0^{(1)}(x_1 r)}{f'(x_1)} - \frac{k^2\alpha_1 \sin(x_1 r_0)\, H_0^{(2)}(x_1 r)}{f'(-x_1)} = \frac{2\alpha_1 k^2 \sin(x_1 r_0)\, J_0(x_1 r)}{f'(x_1)}, \quad \text{where} \quad f'(x) = \frac{\mathrm{d}f(x)}{\mathrm{d}x}. \tag{77}$$

The displacement $w(r, 0)$ now assumes the form

$$w(r,0) = \frac{P\,\mathrm{e}^{\mathrm{i}pt}}{16\mu\pi r_0}\left[\frac{2\pi \mathrm{i} k^2\alpha_1 \sin(x_1 r_0)\, J_0(x_1 r)}{f'(x_1)} + 2\int_k^\infty \frac{\alpha k^2 \sin(x r_0)\, H_0^{(2)}(xr)\,\mathrm{d}x}{f(x)} + 2\int_h^k \frac{k^2\alpha(x^2-\frac{1}{2}k^2)^2 \sin(x r_0)\, H_0^{(2)}(xr)\,\mathrm{d}x}{f(x)\,F(x)}\right], \tag{78}$$

where

$$F(x) = (x^2-\tfrac{1}{2}k^2)^2 + x^2\alpha\beta. \tag{79}$$

Also

$$u(r,0) = \frac{-P\,\mathrm{e}^{\mathrm{i}pt}}{16\mu\pi r_0}\left[\frac{-2\pi x_1(2x_1^2-k^2-\alpha_1\beta_1)\sin(x_1 r_0)\, Y_1(x_1 r)}{f'(x_1)} + \int_h^k \frac{k^2\alpha\beta x(2x^2-k^2)\sin(x r_0)\, H_1^{(2)}(xr)\,\mathrm{d}x}{f(x)\,F(x)}\right]. \tag{80}$$

Because of the contour integration the Cauchy principal value of the integrals in (78) and (80) is implied. A consideration of the three integrals occurring in (78) and (80) will show that because (xr) is positive in the ranges of integration and because of the factors $H_m^{(2)}(xr)$, the integrals are sums of waves diverging from the origin.

However, the first terms in u and w represent standing waves. Now it is to be noticed that if to these solutions the free waves u_0, w_0, equations (59) and (60), are added, then none of the boundary conditions already satisfied are violated in any way, and the result is still a solution. Further, by choosing the amplitude of the waves u_0 and w_0 correctly, it is possible to convert the standing wave part in above expressions into a travelling wave and so satisfy the radial boundary at infinity.

In (59) and (60) put

$$D_1 = \frac{-P 2\pi \mathrm{i} \sin(x_1 r_0)}{16\mu\pi r_0 f'(x_1)}, \tag{81}$$

and

$$u_0 = \frac{P\,\mathrm{e}^{\mathrm{i}pt}\, 2\pi \mathrm{i} \sin(x_1 r_0)\, x_1(2x_1^2-k^2-2\alpha_1\beta_1)\, J_1(x_1 r)}{16\mu\pi r_0 f'(x_1)}, \tag{82}$$

$$w_0 = \frac{-P\,\mathrm{e}^{\mathrm{i}pt}\, 2\pi \mathrm{i} \sin(x_1 r_0)\, k^2\alpha_1 J_0(x_1 r)}{16\mu\pi r_0 f'(x_1)}. \tag{83}$$

Addition of these to (78) and (80) cancels the standing wave part of $w(r, 0)$ and changes that of $u(r, 0)$ to a term containing $\mathrm{i}[J_1(xr) - \mathrm{i}Y_1(xr)] = \mathrm{i}H_1^{(2)}(xr)$, i.e. a diverging wave. All the boundary conditions are now satisfied and the final displacements are given by

$$u(r,0) = \frac{-P\,\mathrm{e}^{\mathrm{i}pt}}{8\mu\pi r_0}\int_0^\infty \frac{x(2x^2-k^2-2\alpha\beta)\sin(x r_0)\, J_1(xr)\,\mathrm{d}x}{f(x)} + \frac{P\,\mathrm{e}^{\mathrm{i}pt}\, 2\pi \mathrm{i} \sin(x_1 r_0)\, x_1(2x_1^2-k^2-2\alpha_1\beta_1)\, J_1(x_1 r)}{16\mu\pi r_0 f'(x_1)}, \tag{84}$$

$$w(r,0) = \frac{P\,\mathrm{e}^{\mathrm{i}pt}}{8\mu\pi r_0}\int_0^\infty \frac{\alpha k^2 \sin(x r_0)\, J_0(xr)\,\mathrm{d}x}{f(x)} - \frac{P\,\mathrm{e}^{\mathrm{i}pt}\, 2\pi \mathrm{i} \sin(x_1 r_0)\, k^2\alpha_1 J_0(x_1 r)}{16\mu\pi r_0 f'(x_1)}, \tag{85}$$

where the Cauchy principal values of the integrals are now understood. It is convenient to change the parameter x and to introduce the following non-dimensional variables:

$$x = k\theta, \quad a = kr, \quad a_0 = kr_0, \quad h/k = \tau, \quad x_1 = k\theta_1; \tag{86}$$

then

$$w(r,0) = \frac{P\,\mathrm{e}^{\mathrm{i}pt}}{8\mu\pi r_0}\int_0^\infty \frac{(\theta^2-\tau^2)^{\frac{1}{2}}\sin(a_0\theta)\,J_0(a\theta)\,\mathrm{d}\theta}{f(\theta)} - \frac{P\,\mathrm{e}^{\mathrm{i}pt}\,2\pi\mathrm{i}\sin(a_0\theta_1)\,(\theta_1^2-\tau^2)^{\frac{1}{2}}\,J_0(a\theta_1)}{16\mu\pi r_0 f'(\theta_1)} \tag{87}$$

and

$$f(\theta) = (\theta^2-\tfrac{1}{2})^2 - \theta^2(\theta^2-\tau^2)^{\frac{1}{2}}(\theta^2-1)^{\frac{1}{2}}.$$

This is the solution of the vertical surface displacement in a homogeneous elastic medium when excited by a vertical stress distribution over the area $0 \leqslant r \leqslant r_0$, equal to the static stress distribution of a rigid circular plate. At low frequencies $w(r,0)$ $(r \leqslant r_0)$, will be approximately constant, but as the frequency increases it will become curved. $w(r,0)$ is now examined in order to estimate the displacement of a rigid plate. This displacement will obviously correspond to some form of average of $w(r,0)$ over $r \leqslant r_0$.

Put

$$w(r,0) = \frac{P\,\mathrm{e}^{\mathrm{i}pt}}{\mu r_0}[F_1(a,a_0,\tau) + \mathrm{i}F_2(a,a_0,\tau)]. \tag{88}$$

F_1 and F_2 are the real and imaginary parts of equation (87). The amplitude of the vibration at any radius is given by

$$A(r,0) = \frac{P}{\mu r_0}(F_1^2+F_2^2)^{\frac{1}{2}}. \tag{89}$$

It is shown later that, in the range of a_0 relevant to the problem, $|F_1|$ and $|F_2|$, as functions of a or r, have a negative gradient when $r \leqslant r_0$, i.e. in this range $|F_1|$ and $|F_2|$ are greatest at the centre and smallest at $r = r_0$. Now $f_1(a_0,\tau)$ and $f_2(a_0,\tau)$ have already been defined as the in-phase and out-of-phase components of displacement of a rigid plate. It is easy to show that $|f_1(a_0,\tau)|$ is greater than $|F_1(a_0,a_0,\tau)|$ and $|f_2(a_0,\tau)|$ is greater than $|F_2(a_0,a_0,\tau)|$.

Consider figure 2, illustrating $|F_1|$, $|F_2|$ and $(F_1^2+F_2^2)^{\frac{1}{2}}$. As a_0 increases, a greater part of the effective wavelength is bridged by point a_0 or r_0 and it moves out to some point z say.

In order to convert the case we have solved to that of a rigid plate, forces p_r and q_r must be superimposed on the stress distribution assumed, as indicated in figure 2, in order to make $(F_1^2+F_2^2)^{\frac{1}{2}}$ constant, i.e. a rigid state. As the total force on the area is to be constant and equal to P, $\Sigma p_r + \Sigma q_r = 0$ and the Σp_r will be outside some point X and Σq_r inside X. Then, whatever the distribution of Σp_r and Σq_r, Σp_r will be closer to the point $r = r_0$ than Σq_r, so that the depression of $r = r_0$ due to Σp_r is greater than the lift of $r = r_0$ due to Σq_r. Similarly, the opposite is true for the point $r = 0$. This means that deflexion of a rigid plate lies between the deflexions at the centre and periphery, i.e.

$$|F_{1,2}(a_0,a_0,\tau)| < |f_{1,2}(a_0,\tau)| < |F_{1,2}(0,a_0,\tau)|. \tag{90}$$

However, it is possible to find a much lower upper limit than $|F_{1,2}(0,a_0,\tau)|$. Apply the reciprocal theorem of Rayleigh (1944, p. 153) to the two sets of stresses and displacements, i.e. the static stress distribution giving the displacements $P\,\mathrm{e}^{\mathrm{i}pt}(F_1+\mathrm{i}F_2)/\mu r_0$ and the rigid plate stress distribution giving the displacement,

$$P\,\mathrm{e}^{\mathrm{i}pt}[f_1+\mathrm{i}f_2]/\mu r_0.$$

Replacing the summation by an integral over the circle $0 \leqslant r \leqslant r_0$, we find that

$$\frac{P^2}{\mu r_0}[f_1(a_0,\tau)+\mathrm{i}f_2(a_0,\tau)] = \int_0^{r_0} \frac{2\pi r P}{2\pi r_0 (r_0^2-r^2)^{\frac{1}{2}}} \frac{P}{\mu r_0}[F_1(a,a_0,\tau)+\mathrm{i}F_2(a,a_0,\tau)]\,\mathrm{d}r$$
$$+\Sigma p_r \frac{P}{\mu r_0}[F_1(a,a_0,\tau)+\mathrm{i}F_2(a,a_0,\tau)]+\Sigma q_r \frac{P}{\mu r_0}[F_1(a,a_0,\tau)+\mathrm{i}F_2(a,a_0,\tau)]. \quad (91)$$

It follows from figure 2 that the sum of the last two terms in (91) is negative. Hence

$$|f_{1,2}(a_0,\tau)| < \left| \int_0^{r_0} \frac{rF_{1,2}(a,a_0,\tau)\,\mathrm{d}r}{r_0(r_0^2-r^2)^{\frac{1}{2}}} \right|. \quad (92)$$

This last term provides a closer upper limit to $f_{1,2}(a_0,\tau)$ and is an average of the displacements over $r \leqslant r_0$ formed by weighting the displacement according to the force acting there. It will be shown, by integrating the results, that this upper limit lies close to the lower one, i.e. the displacement at $r = r_0$. The 'average value' is now taken to mean this kind of average.

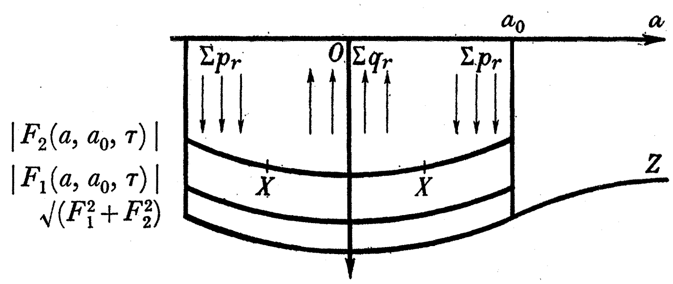

FIGURE 2

It will be designated by U_a, V_a, W_a and the peripheral displacement by U_p, V_p, W_p:

$$W_a = \int_0^{r_0} \frac{r\,\mathrm{d}r}{r_0(r_0^2-r^2)^{\frac{1}{2}}} \left\{ \frac{P\,\mathrm{e}^{\mathrm{i}pt}}{8\mu\pi r_0} \int_0^\infty \frac{k^2\alpha \sin(xr_0)\,J_0(xr)\,\mathrm{d}x}{f(x)} - \frac{P\,\mathrm{e}^{\mathrm{i}pt}\,2\pi\mathrm{i}\sin(x_1 r_0)\,k^2\alpha_1 J_0(x_1 r)}{16\mu\pi r_0 f'(x_1)} \right\}. \quad (93)$$

The infinite integral may be shown to be absolutely convergent and may be integrated under the integral sign:

$$\int_0^{r_0} \frac{rJ_0(xr)\,\mathrm{d}r}{(r_0^2-r^2)^{\frac{1}{2}}} = \int_0^{\frac{1}{2}\pi} r_0 \sin\phi\, J_0(xr_0\sin\phi)\,\mathrm{d}\phi$$
$$= \frac{\sin(xr_0)}{x}, \quad (94)$$

i.e. Sonine's equation (24). The displacements follow as

$$W_a = \frac{P\,\mathrm{e}^{\mathrm{i}pt}}{8\mu\pi r_0} \int_0^\infty \frac{(\theta^2-\tau^2)^{\frac{1}{2}} \sin^2(a_0\theta)\,\mathrm{d}\theta}{f(\theta)\,a_0\theta} - \frac{P\,\mathrm{e}^{\mathrm{i}pt}\,2\pi\mathrm{i}(\theta_1^2-\tau^2)^{\frac{1}{2}}\sin^2(a_0\theta_1)}{16\mu\pi r_0 f'(\theta_1)\,a_0\theta_1}$$
$$\equiv \frac{P\,\mathrm{e}^{\mathrm{i}pt}}{\mu r_0}[f_{1a}(a_0,\tau)+\mathrm{i}f_{2a}(a_0,\tau)], \quad (95)$$

$$W_p = \frac{P\,\mathrm{e}^{\mathrm{i}pt}}{8\mu\pi r_0} \int_0^\infty \frac{(\theta^2-\tau^2)^{\frac{1}{2}} \sin(a_0\theta)\,J_0(a_0\theta)\,\mathrm{d}\theta}{f(\theta)} - \frac{P\,\mathrm{e}^{\mathrm{i}pt}\,2\pi\mathrm{i}(\theta_1^2-\tau^2)^{\frac{1}{2}}\sin(a_0\theta_1)\,J_0(a_0\theta_1)}{16\mu\pi r_0 f'(\theta_1)}$$
$$\equiv \frac{P\,\mathrm{e}^{\mathrm{i}pt}}{\mu r_0}[f_{1p}(a_0,\tau)+\mathrm{i}f_{2p}(a_0,\tau)]. \quad (96)$$

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(iii) *Evaluation of W_a and W_p.* The infinite integrals occurring in W_a and W_p contain both a real and an imaginary part, i.e. they contain $f_{1a,p}$ and the part of $f_{2a,p}$ which corresponds to energy propagated and lost spatially. The residue or free wave term, also imaginary, accounts for the rest of $\mathrm{i}f_{2a,p}$ and corresponds to the energy lost in Rayleigh surface waves.

Put
$$I=\int_0^\infty \frac{(\theta^2-\tau^2)^{\frac{1}{2}}\,M(a\theta)\sin(a_0\theta)\,\mathrm{d}\theta}{[(\theta^2-\frac{1}{2})^2-\theta^2(\theta^2-\tau^2)^{\frac{1}{2}}(\theta^2-1)^{\frac{1}{2}}}=I'+I'', \tag{97}$$

where $M(a,\theta)=J_0(a\theta)$ or $\dfrac{\sin(a_0\theta)}{a_0\theta}$,

$$I'=\int_0^\infty \frac{(\theta^2-\tau^2)^{\frac{1}{2}}(\theta^2-\frac{1}{2})^2\,M(a\theta)\sin(a_0\theta)\,\mathrm{d}\theta}{[(\theta^2-\frac{1}{2})^4-\theta^4(\theta^2-\tau^2)(\theta^2-1)}, \tag{98}$$

$$I''=\int_0^\infty \frac{\theta^2(\theta^2-\tau^2)(\theta^2-1)^{\frac{1}{2}}\,M(a\theta)\sin(a_0\theta)\,\mathrm{d}\theta}{[(\theta^2-\frac{1}{2})^4-\theta^4(\theta^2-\tau^2)(\theta^2-1)]}. \tag{99}$$

The integrals
$$K'=\int_{-\infty}^\infty \frac{(\theta^2-\tau^2)^{\frac{1}{2}}(\theta^2-\frac{1}{2})^2\,M(a\theta)\,\mathrm{e}^{\mathrm{i}a_0\theta}\,\mathrm{d}\theta}{[(\theta^2-\frac{1}{2})^4-\theta^4(\theta^2-\tau^2)(\theta^2-1)]}, \tag{100}$$

$$K''=\int_{-\infty}^\infty \frac{\theta^2(\theta^2-\tau^2)(\theta^2-1)^{\frac{1}{2}}\,M(a\theta)\,\mathrm{e}^{\mathrm{i}a_0\theta}\,\mathrm{d}\theta}{[(\theta^2-\frac{1}{2})^4-\theta^4(\theta^2-\tau^2)(\theta^2-1)]} \tag{101}$$

are integrated around an infinite semicircle in the upper half-plane and the sign of the radical on the left of the branch points, $\theta=-1$, $\theta=-\tau$, is changed to make the integrand analytic. It can be shown that the integral around the semicircle vanishes if $a\leqslant a_0$. The following expansion can thus be used to determine displacements inside the circle $r\leqslant r_0$, whereas the previous expansion is only true for $r\geqslant r_0$, i.e. displacements outside the circle. There are six poles at the points $\pm\theta_1$, $\pm\theta_2$, $\pm\theta_3$ and R'_r is the sum of the residues of K' at $\pm\theta_r$. This contour integration gives

$$I'=\tfrac{1}{2}\pi\sum_r R'_r-\int_0^\tau \frac{(\theta^2-\frac{1}{2})^2(\tau^2-\theta^2)^{\frac{1}{2}}\,M(a\theta)\cos(a_0\theta)\,\mathrm{d}\theta}{[(\theta^2-\frac{1}{2})^4-\theta^4(\theta^2-\tau^2)(\theta^2-1)]}$$
$$+\mathrm{i}\int_0^\tau \frac{(\theta^2-\frac{1}{2})^2(\tau^2-\theta^2)^{\frac{1}{2}}\,M(a\theta)\sin(a_0\theta)\,\mathrm{d}\theta}{[(\theta^2-\frac{1}{2})^4-\theta^4(\theta^2-\tau^2)(\theta^2-1)]}, \tag{102}$$

$$I''=\tfrac{1}{2}\pi\sum_r R''_r-\int_0^1 \frac{\theta^2(\theta^2-\tau^2)(1-\theta^2)^{\frac{1}{2}}\,M(a\theta)\cos(a_0\theta)\,\mathrm{d}\theta}{[(\theta^2-\frac{1}{2})^4-\theta^4(\theta^2-\tau^2)(\theta^2-1)]}$$
$$+\mathrm{i}\int_0^1 \frac{\theta^2(\theta^2-\tau^2)(1-\theta^2)^{\frac{1}{2}}\,M(a\theta)\sin(a_0\theta)\,\mathrm{d}\theta}{[(\theta^2-\frac{1}{2})^4-\theta^4(\theta^2-\tau^2)(\theta^2-1)]} \tag{103}$$

and $I=I'+I''$
$$=\tfrac{1}{2}\pi\sum_r(R'_r+R''_r)-\int_0^\tau \frac{(\tau^2-\theta^2)^{\frac{1}{2}}\,M(a\theta)\cos(a_0\theta)\,\mathrm{d}\theta}{f(\theta)}$$
$$-\int_\tau^1 \frac{\theta^2(\theta^2-\tau^2)(1-\theta^2)^{\frac{1}{2}}\,M(a\theta)\cos(a_0\theta)\,\mathrm{d}\theta}{[(\theta^2-\frac{1}{2})^4-\theta^4(\theta^2-\tau^2)(\theta^2-1)]}$$
$$+\mathrm{i}\int_0^\tau \frac{(\tau^2-\theta^2)^{\frac{1}{2}}\,M(a\theta)\sin(a_0\theta)\,\mathrm{d}\theta}{f(\theta)}+\mathrm{i}\int_\tau^1 \frac{\theta^2(\theta^2-\tau^2)(1-\theta^2)^{\frac{1}{2}}\,M(a\theta)\sin(a_0\theta)\,\mathrm{d}\theta}{[(\theta^2-\frac{1}{2})^4-\theta^4(\theta^2-\tau^2)(\theta^2-1)]}, \tag{104}$$

if $a\leqslant a_0$.

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We are not interested in values of a_0 greater than 1·5, and the functions $M(a\theta)\cos(a_0\theta)$ and $M(a\theta)\sin(a_0\theta)$ may be expressed as series. It is necessary, then, to evaluate the above integrals with θ^n in place of $M(a\theta)\cos(a_0\theta)$, etc., for a sufficient number of values of n. These integrals may be integrated by rationalization but were, in fact, integrated graphically. Rotation of the plate about a horizontal axis gives an exactly similar integral but with a different series occurring in place of $M(a\theta)\cos(a_0\theta)$. By putting in the respective coefficients both these cases may be integrated together. The residues are given as follows. If θ_r lies outside τ or 1 then the sign must be changed in the radical of the residue at $-\tau$ or -1:

$$\theta_r > \tau, \quad R'_r = \frac{\cos(a_0\theta_r)(\theta_r^2-\frac{1}{2})^2(\theta_r^2-\tau^2)^{\frac{1}{2}}M(a\theta)}{\theta_r(\theta_r^2-\theta_2^2)(\theta_r^2-\theta_3^2)(\tau^2-1)}, \tag{105}$$

$$\theta_r < \tau, \quad R'_r = \frac{\mathrm{i}\sin(a_0\theta_r)(\theta_r^2-\frac{1}{2})^2(\theta_r^2-\tau^2)M(a\theta)}{\theta_r(\theta_r^2-\theta_2^2)(\theta_r^2-\theta_3^2)(\tau^2-1)}, \tag{106}$$

$$\theta_r > 1, \quad R''_r = \frac{\cos(a_0\theta_r)\,\theta_r^2(\theta_r^2-\tau^2)(\theta_r^2-1)^{\frac{1}{2}}M(a\theta)}{\theta_r(\theta_r^2-\theta_2^2)(\theta_r^2-\theta_3^2)(\tau^2-1)}, \tag{107}$$

$$\theta_r < 1, \quad R''_r = \frac{\mathrm{i}\sin(a_0\theta_r)\,\theta_r^2(\theta_r^2-\tau^2)(\theta_r^2-1)^{\frac{1}{2}}M(a\theta)}{\theta_r(\theta_r^2-\theta_2^2)(\theta_r^2-\theta_3^2)(\tau^2-1)}. \tag{108}$$

The values of θ_r^2 are easily determined as the roots of the cubic,

$$(x-\tfrac{1}{2})^4 - x^2(x-\tau^2)(x-1) \equiv x^3(\tau^2-1) + x^2(\tfrac{3}{2}-\tau^2) - \tfrac{1}{2}x + \tfrac{1}{16}, \tag{109}$$

and the roots are

$$\left.\begin{aligned} \tau^2 &= 0{\cdot}500, & x &= 0{\cdot}500, \quad 0{\cdot}192, \quad 1{\cdot}308,\\ \tau^2 &= 0, & x &= 0{\cdot}203 \pm \mathrm{i}0{\cdot}128, \quad 1{\cdot}095,\\ \tau^2 &= 0{\cdot}333, & x &= 0{\cdot}250, \quad 0{\cdot}318, \quad 1{\cdot}184. \end{aligned}\right\} \tag{110}$$

The last root quoted in each of the three cases of τ^2 is recognized as the square of the roots quoted previously for $f(x) = 0$. The computation indicated gives series for the integrals and the total values of $f_{1a,p}$ and $f_{2a,p}$ as functions of a_0 and τ are exhibited in figures 4 and 5. This series integration was used only for the value of Poisson's ratio used in the experimental work, i.e. $\nu = 0$, $\tau^2 = \frac{1}{2}$. The curves f_{1a}, f_{2a} for the two other values of Poisson's ratio were integrated graphically, directly:

$$K'_n = \int_0^{0{\cdot}707} \frac{(\frac{1}{2}-\theta^2)^{\frac{1}{2}}\theta^n\,\mathrm{d}\theta}{f(\theta)}, \tag{111}$$

$$K''_n = \int_{0{\cdot}707}^{1} \frac{\theta^2(\theta^2-\frac{1}{2})(1-\theta^2)^{\frac{1}{2}}\theta^n\,\mathrm{d}\theta}{[(\theta^2-\frac{1}{2})^4-\theta^4(\theta^2-\frac{1}{2})(\theta^2-1)]}. \tag{112}$$

TABLE 1

n	K'_n	K''_n
0	2·047	0·678
1	0·729	0·566
2	0·344	0·479
3	0·182	0·408
4	0·103	0·353
5	0·061	0·306
6	0·043	0·267
7	0·030	0·233
8	0·021	0·209

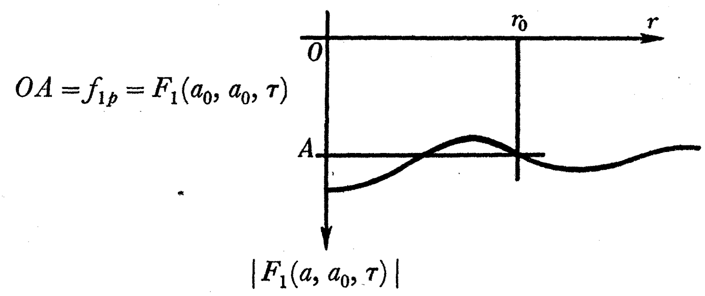

FIGURE 3

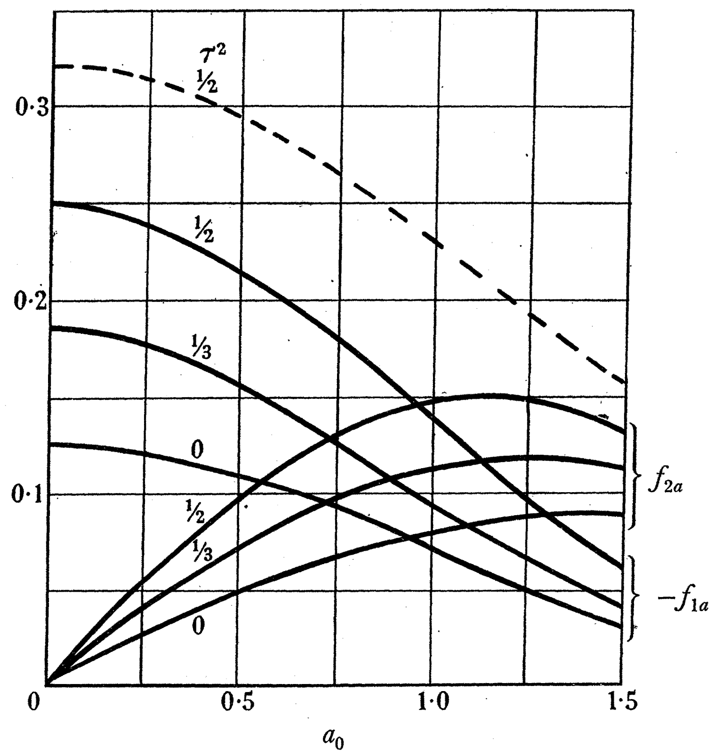

FIGURE 4. Semi-infinite space, vertical translation. ---, Reissner's estimation of f_1 for $\tau^2 = \frac{1}{2}$.

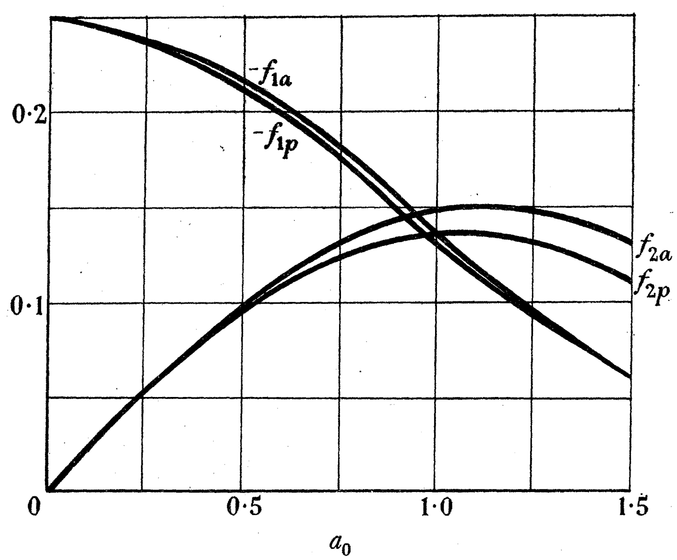

FIGURE 5. Limits for the functions f_1, f_2; semi-infinite space, vertical translation; $\tau^2 = \frac{1}{2}$.

Expand
$$\frac{\sin^2(a_0\theta)}{a_0\theta},\quad \frac{\sin(a_0\theta)\cos(a_0\theta)}{a_0\theta},$$
$$J_0(a_0\theta)\cos(a_0\theta),\quad J_0(a_0\theta)\sin(a_0\theta)$$
as series, and then
$$\int_0^{0\cdot707}\frac{(\frac{1}{2}-\theta^2)^{\frac{1}{2}}\sin(a_0\theta)\cos(a_0\theta)\,d\theta}{f(\theta)\,a_0\theta}+\int_{0\cdot707}^1\frac{\theta^2(\theta^2-\frac{1}{2})\,(1-\theta^2)^{\frac{1}{2}}\sin(a_0\theta)\cos(a_0\theta)\,d\theta}{[(\theta^2-\frac{1}{2})^4-\theta^4(\theta^2-\frac{1}{2})\,(\theta^2-1)]\,a_0\theta}$$
$$=2\cdot725-0\cdot550a_0^2+0\cdot0606a_0^4-0\cdot0039a_0^6+0\cdot0002a_0^8-\ldots, \quad (113)$$
$$\int_0^{0\cdot707}\frac{(\frac{1}{2}-\theta^2)^{\frac{1}{2}}\sin^2(a_0\theta)\,d\theta}{f(\theta)\,a_0\theta}+\int_{0\cdot707}^1\frac{\theta^2(\theta^2-\frac{1}{2})\,(1-\theta^2)^{\frac{1}{2}}\sin^2(a_0\theta)\,d\theta}{[(\theta^2-\frac{1}{2})^4-\theta^4(\theta^2-\frac{1}{2})\,(\theta^2-1)]\,a_0\theta}$$
$$=1\cdot295a_0-0\cdot196a_0^3+0\cdot0163a_0^5-0\cdot00082a_0^7+\ldots, \quad (114)$$
$$\int_0^{0\cdot707}\frac{(\frac{1}{2}-\theta^2)^{\frac{1}{2}}J_0(a_0\theta)\cos(a_0\theta)\,d\theta}{f(\theta)}+\int_{0\cdot707}^1\frac{\theta^2(\theta^2-\frac{1}{2})\,(1-\theta^2)^{\frac{1}{2}}J_0(a_0\theta)\cos(a_0\theta)\,d\theta}{[(\theta^2-\frac{1}{2})^4-\theta^4(\theta^2-\frac{1}{2})\,(\theta^2-1)]}$$
$$=2\cdot725-0\cdot618a_0^2+0\cdot083a_0^4-0\cdot006a_0^6+0\cdot0003a_0^8+\ldots, \quad (115)$$
$$\int_0^{0\cdot707}\frac{(\frac{1}{2}-\theta^2)^{\frac{1}{2}}\sin(a_0\theta)\,J_0(a_0\theta)\,d\theta}{f(\theta)}+\int_{0\cdot707}^1\frac{\theta^2(\theta^2-\frac{1}{2})\,(1-\theta^2)^{\frac{1}{2}}\sin(a_0\theta)\,J_0(a_0\theta)\,d\theta}{[(\theta^2-\frac{1}{2})^4-\theta^4(\theta^2-\frac{1}{2})\,(\theta^2-1)]}$$
$$=1\cdot295a_0-0\cdot256a_0^3+0\cdot0241a_0^5-0\cdot0012a_0^7+\ldots, \quad (116)$$

(iv) *Slope of surface*. The upper and lower limits are derived on a basis of $|F_1(a,a_0,\tau)|$ and $|F_2(a,a_0,\tau)|$ having a negative slope with respect to a or r. The method of integration of F_1 and F_2 is true for $a\leqslant a_0$ and was used to investigate their slopes for increasing a_0. $\partial F_1/\partial a$ changes sign at $a_0 \doteqdot 1\cdot41$ for $\tau^2=\frac{1}{2}$ and $\partial F_2/\partial a$ changes sign at $a_0\doteqdot 2\cdot50$.

Values of a_0 greater than this are not needed and the bounds of f_1, f_2 are correct up to $a_0=1\cdot41$, but it may be fairly safely assumed that the value f_{1a}, f_{2a} is a close approximation to f_1, f_2 for much higher values of a_0.

It is seen from the graphs of f_{1a}, f_{2a} (figure 5) that at a value of $a_0\doteqdot 1\cdot50$, f_{1a} and f_{1p} have become equal, indicating that the shape of $F_1(a,a_0,\tau)$ as a function of a is now of the form shown in figure 3.

The change of slope occurring has made it possible for $F_1(a_0,a_0,\tau)\equiv f_{1p}(a_0,\tau)$ to increase relative to $f_{1a}(a_0,\tau)$ so that at $a_0\doteqdot 1\cdot50$ they become equal. At higher values of a_0, $|f_{1p}(a_0,\tau)|$ is greater than $|f_{1a}(a_0,\tau)|$. The computed values of the displacement functions are shown in figures 4 and 5. Figures 6, 7 and 8 show amplitude, phase, and average power input curves when the plate is loaded with a mass.

Elastic stratum

(*a*) *Dynamic displacements*

By compounding equations (50) to (53) with the corresponding ones involving $-\alpha$, $-\beta$ the following are obtained as solutions to the elastic wave equation:
$$w=\left[\frac{-2A\alpha}{h^2}\sinh(\alpha z)+\frac{2Cx^2}{k^2}\sinh(\beta z)\right]J_0(xr)\,e^{ipt}, \quad (117)$$
$$u=\left[\frac{2Ax}{h^2}\cosh(\alpha z)-\frac{2C\beta x}{k^2}\cosh(\beta z)\right]J_1(xr)\,e^{ipt}, \quad (118)$$
$$\widehat{zz}=2\mu\left[\frac{-2A}{h^2}(x^2-\tfrac{1}{2}k^2)\cosh(\alpha z)+\frac{2C\beta x^2}{k^2}\cosh(\beta z)\right]J_0(xr)\,e^{ipt}, \quad (119)$$
$$\widehat{zr}=2\mu\left[\frac{-2A\alpha x}{h^2}\sinh(\alpha z)+\frac{2Cx}{k^2}(x^2-\tfrac{1}{2}k^2)\sinh(\beta z)\right]J_1(xr)\,e^{ipt}. \quad (120)$$

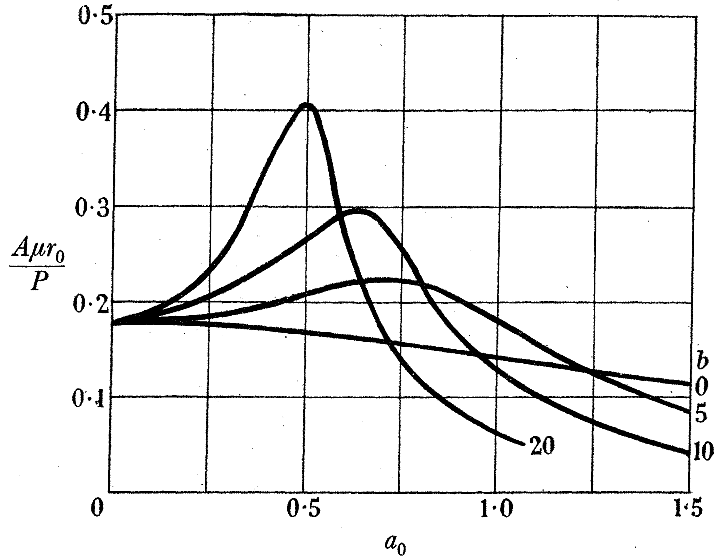

FIGURE 6. Amplitude curves, semi-infinite space, vertical translation; $\tau^2 = \frac{1}{3}$.

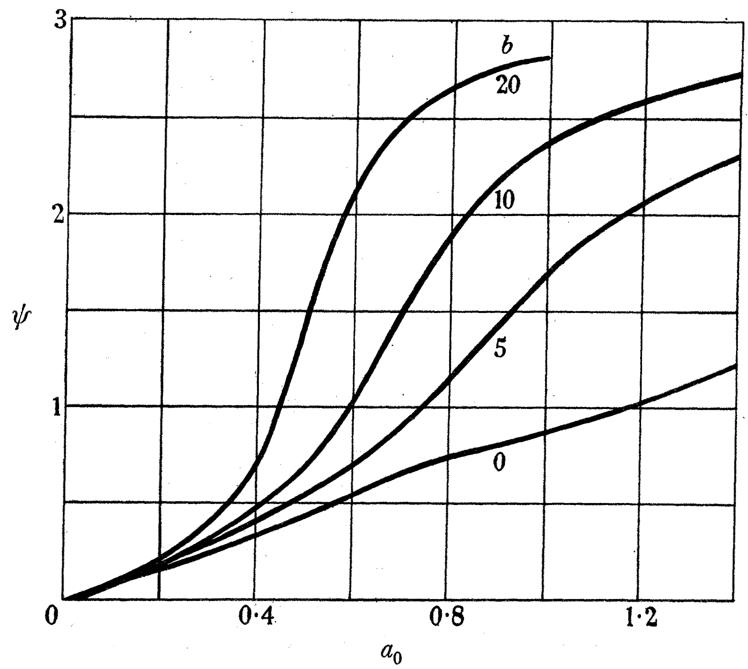

FIGURE 7. Phase angle, semi-infinite space, vertical translation.

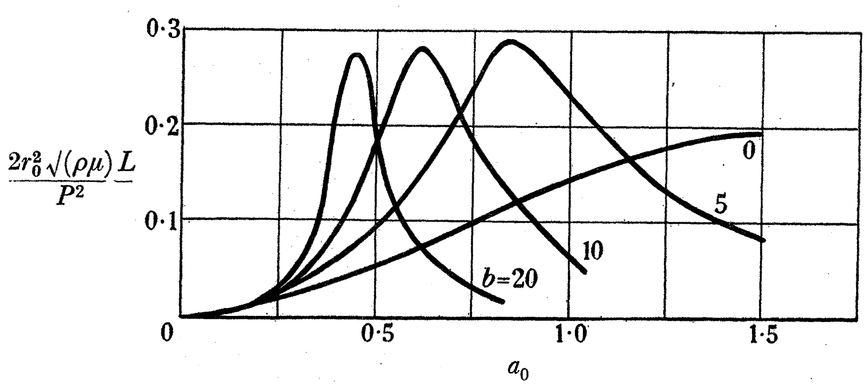

FIGURE 8. Average power input, semi-infinite space, vertical translation; $\tau^2 = \frac{1}{3}$.

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If $z = 0$ is chosen to represent the lower surface of the stratum, the conditions $z = 0$, $w = \widehat{zr} = 0$, are already satisfied, i.e. the problem where the elastic stratum may slide on a rigid foundation but has its vertical displacements restrained there. The upper free surface is taken as $z = \delta$.

(i) *Free waves.* Equating $\widehat{zz} = \widehat{zr} = 0$, $z = \delta$, gives the following frequency equation obtained by Marguerre (1933):

$$l_1(x) = (x^2 - \tfrac{1}{2}k^2)^2 \cosh(\alpha\delta)\sinh(\beta\delta) - \alpha\beta x^2 \cosh(\beta\delta)\sinh(\alpha\delta) = 0. \tag{121}$$

This equation only has real roots above certain values of h, k, depending on the stratum depth δ and the number of these roots depends on h, k, δ, τ. Change the variable x to $k\theta$, and

$$l_1(\theta) = (\theta^2 - \tfrac{1}{2})^2 \cosh[\gamma(\theta^2-\tau^2)^{\frac{1}{2}}]\sinh[\gamma(\theta^2-1)^{\frac{1}{2}}] - \theta^2(\theta^2-\tau^2)^{\frac{1}{2}}(\theta^2-1)^{\frac{1}{2}}\cosh[\gamma(\theta^2-1)^{\frac{1}{2}}]\sinh[\gamma(\theta^2-\tau^2)^{\frac{1}{2}}] \tag{122}$$

and $$\gamma = k\delta = kr_0\delta/r_0 = a_0 R.$$

This function may only be satisfactorily examined graphically. When $\tau^2 = \frac{1}{2}$ there are only real roots if $\gamma > \pi/\sqrt{2}$. If $\tau^2 = \frac{1}{3}$ there are real roots for any γ. If the nth real root is denoted by x_n the free waves are given by

$$\left.\begin{aligned} u_0(r,z) &= -\sum_n D_n e^{ipt} x_n[(x_n^2 - \tfrac{1}{2}k^2)\cosh(\alpha_n z)\sinh(\beta_n\delta) - \alpha_n\beta_n\sinh(\alpha_n\delta)\cosh(\beta_n z)]\,J_1(x_n r),\\ w_0(r,z) &= \sum_n D_n e^{ipt}\alpha_n[(x_n^2-\tfrac{1}{2}k^2)\sinh(\alpha_n z)\sinh(\beta_n\delta) - x_n^2\sinh(\beta_n z)\sinh(\alpha_n\delta)]\,J_0(x_n r). \end{aligned}\right\} \tag{123}$$

(ii) *Forced vibrations.* It does not appear possible to calculate the static stress distribution under the plate attached to the stratum surface by any simple means. The solution may be approximated to as closely as desired by continued solution of an infinite set of simultaneous equations but the computation involved is excessive. By applying the static stress distribution found for the plate on a semi-infinite space which, for not too shallow strata, will be a close approximation it is possible to examine the main aspects of the case. As in the semi-infinite case the following tentative expressions are obtained for the displacements and, as before, they must be examined in respect to the radial boundary conditions at infinity:

$$u(r,\delta) = \frac{-P e^{ipt}}{8\mu\pi r_0}\int_0^\infty \frac{x[(2x^2-k^2)\cosh(\alpha\delta)\sinh(\beta\delta) - 2\alpha\beta\cosh(\beta\delta)\sinh(\alpha\delta)]\sin(xr_0)\,J_1(xr)\,dx}{l_1(x)}, \tag{124}$$

$$w(r,\delta) = \frac{-P e^{ipt}}{8\mu\pi r_0}\int_0^\infty \frac{\alpha k^2\sinh(\alpha\delta)\sinh(\beta\delta)\sin(xr_0)\,J_0(xr)\,dx}{l_1(x)}. \tag{125}$$

Make the substitution $J_m(xr) = \frac{1}{2}[H_m^{(1)}(xr) + H_m^{(2)}(xr)]$ and integrate around an infinite semicircle in the upper half-plane, changing the signs of α and β at the branch points in order to make the integrand analytic. However, it is noted that the integrand is even with respect to α and β, and changing their signs does not alter the value of the integrand. The integral around the semicircle vanishes, as before, for $r \geq r_0$. The function $l_1(x)$, besides its N real roots, has an infinite number of complex roots denoted by $\xi_v = \zeta_v + i\eta_v$. η_v is to be positive, i.e. we are only interested in poles in the upper half-plane. This contour integration then gives

$$u(r,\delta) = \frac{-P\,\mathrm{e}^{\mathrm{i}pt}}{16\mu\pi r_0}\int_{-\infty}^{\infty}\frac{x[(2x^2-k^2)\cosh(\alpha\delta)\sinh(\beta\delta)-2\alpha\beta\cosh(\beta\delta)\sinh(\alpha\delta)]\sin(xr_0)\,H_1^{(1)}(xr)\,\mathrm{d}x}{l_1(x)}$$

$$= 2\pi\mathrm{i}\sum_v R^{(u)}_{\xi_v} + \pi\mathrm{i}\sum_N R^{(u)}_{x_n}, \tag{126}$$

$$w(r,\delta) = \frac{-P\,\mathrm{e}^{\mathrm{i}pt}}{16\mu\pi r_0}\int_{-\infty}^{\infty}\frac{k^2\alpha\sinh(\alpha\delta)\sinh(\beta\delta)\sin(xr_0)\,H_0^{(1)}(xr)\,\mathrm{d}x}{l_1(x)}$$

$$= 2\pi\mathrm{i}\sum_{v=1}^{\infty} R^{(w)}_{\xi_v} + \pi\mathrm{i}\sum_{n=1}^{N} R^{(w)}_{x_n}. \tag{127}$$

As before, this expression gives the Cauchy principal value of the integral

$$R^{(w)}_{\xi_v} = \frac{-P\,\mathrm{e}^{\mathrm{i}pt}k^2\alpha_v\sinh(\alpha_v\delta)\sinh(\beta_v\delta)\sin(\xi_v r_0)\,H_0^{(1)}(\xi_v r)}{16\mu\pi r_0\,l_1^1(\xi_v)}, \tag{128}$$

$$R^{(w)}_{\pm x_n} = \frac{-P\,\mathrm{e}^{\mathrm{i}pt}k^2\alpha_n\sinh(\alpha_n\delta)\sinh(\beta_n\delta)\sin(x_n r_0)\,2\mathrm{i}Y_0(x_n r)}{16\mu\pi r_0\,l_1^1(x_n)}, \tag{129}$$

$$R^{(u)}_{\xi_v} = \frac{-P\,\mathrm{e}^{\mathrm{i}pt}\xi_v[(2\xi_v^2-k^2)\cosh(\alpha_v\delta)\sinh(\beta_v\delta)-2\alpha_v\beta_v\cosh(\beta_v\delta)\sinh(\alpha_v\delta)]\sin(\xi_v r_0)\,H_1^{(1)}(\xi_v r)}{16\mu\pi r_0\,l_1^1(\xi_v)}, \tag{130}$$

$$R^{(u)}_{\pm x_n} = \frac{-P\,\mathrm{e}^{\mathrm{i}pt}x_n[(2x_n^2-k^2)\cosh(\alpha_n\delta)\sinh(\beta_n\delta)-2\alpha_n\beta_n\cosh(\beta_n\delta)\sinh(\alpha_n\delta)]\sin(x_n r_0)\,2\mathrm{i}Y_1(x_n r)}{16\mu\pi r_0\,l_1^1(x_n)}. \tag{131}$$

Consider the term $$\mathrm{e}^{\mathrm{i}pt}H_0^{(1)}(\xi_v r) = \mathrm{e}^{\mathrm{i}pt}H_0^{(1)}[(\zeta_v+\mathrm{i}\eta_v)\,r], \tag{132}$$

which, as r becomes large, tends to

$$\mathrm{e}^{\mathrm{i}pt}\left(\frac{2}{\pi(\zeta_v+\mathrm{i}\eta_v)\,r}\right)^{\frac{1}{2}}\mathrm{e}^{\mathrm{i}[(\zeta_v+\mathrm{i}\eta_v)r-\frac{1}{4}\pi]} = \left(\frac{2}{\pi(\zeta_v+\mathrm{i}\eta_v)\,r}\right)^{\frac{1}{2}}\mathrm{e}^{-\eta_v r}\,\mathrm{e}^{\mathrm{i}(\zeta_v r+pt)}\,\mathrm{e}^{-\frac{1}{4}\mathrm{i}\pi}.$$

Depending whether ζ_v is positive or negative this term represents a wave travelling away from or towards the origin. In the upper half-plane η_v is positive and the term is thus exponentially damped and is not significant at large distances. $R^{(w)}_{\xi_v}$ is recognized as an exponentially damped travelling wave. Similar remarks apply to $R^{(u)}_{\xi_v}$. Because of the factors $Y_0(x_n r)$ and $Y_1(x_n r)$ $R^{(w)}_{\pm x_n}$ and $R^{(u)}_{\pm x_n}$ are recognized as standing waves decreasing with respect to r in the order of $r^{-\frac{1}{2}}$. At large distances these standing waves become predominant. As in the semi-infinite case, free waves u_0 and w_0 must be added to make the displacements, at a large distance, a travelling wave. At $z=\delta$ the free waves are

$$u_0(r,\delta) = -\sum_n D_n\,\mathrm{e}^{\mathrm{i}pt}x_n[(x_n^2-\tfrac{1}{2}k^2)\cosh(\alpha_n\delta)\sinh(\beta_n\delta)-\alpha_n\beta_n\sinh(\alpha_n\delta)\cosh(\beta_n\delta)]\,J_1(x_n r), \tag{133}$$

$$w_0(r,\delta) = -\sum_n D_n\,\mathrm{e}^{\mathrm{i}pt}\alpha_n\tfrac{1}{2}k^2\sinh(\alpha_n\delta)\sinh(\beta_n\delta)\,J_0(x_n r). \tag{134}$$

If the coefficient is chosen as $$D_n = \frac{-2\pi\mathrm{i}P\sin(x_n r_0)}{8\mu\pi r_0\,l_1^1(x_n)}, \tag{135}$$

the standing waves $R^{(u)}_{\pm x_n}$, $R^{(w)}_{\pm x_n}$ are converted into waves containing the terms $H_1^{(2)}(x_n r)$, $H_0^{(2)}(x_n r)$, i.e. waves travelling from the origin. The full solution to the problem is given by the Cauchy principal value of the integrals in equations (126) and (127), together with the free waves of equations (133), (134) and (135). The 'average displacement' over $r \leqslant r_0$ is obtained by replacing

$$J_0(xr) \quad \text{by} \quad \sin(xr_0)/xr_0.$$

However, it is not proposed to evaluate this integral. Enough of the complex roots ξ_v conceivably could be evaluated by approximating processes, but the work would be excessive. The integral occurring in $w(r,\delta)$ was partly evaluated numerically, and this evaluation revealed that, unlike the semi-infinite case, resonances can occur. Perhaps this is to be expected, but it will be shown that, in the cases of rotation of the plate where it is just as much to be expected, it does not occur.

Consider the integral, the principal value of which occurs in $w(r,\delta)$, and write it as

$$I_1 = \int_0^\infty \frac{(\theta^2-\tau^2)^{\frac{1}{2}} \sin(a_0\theta)\, J_0(a\theta)\, d\theta}{[(\theta^2-\frac{1}{2})^2 \coth\{\gamma(\theta^2-\tau^2)^{\frac{1}{2}}\} - \theta^2(\theta^2-\tau^2)^{\frac{1}{2}}(\theta^2-1)^{\frac{1}{2}} \coth\{\gamma(\theta^2-1)^{\frac{1}{2}}\}]}. \tag{136}$$

If γ is chosen so that
$$\coth\{\gamma(-\tau^2)^{\frac{1}{2}}\} = 0, \quad \text{i.e.} \tag{137}$$

$$\gamma = \frac{(2n-1)\pi}{2\tau}, \tag{138}$$

then, when θ is small,

$$\begin{aligned} \coth\{\gamma(\theta^2-\tau^2)^{\frac{1}{2}}\} &= -\mathrm{i}\cot\left[\gamma\tau\left(1-\frac{\theta^2}{2\tau^2}\right)\right] \\ &= -\mathrm{i}\cot\left[\frac{(2n-1)\pi}{2}\left(1-\frac{\theta^2}{2\tau^2}\right)\right] \\ &= -\mathrm{i}\frac{(2n-1)\pi}{2}\frac{\theta^2}{2\tau^2} \end{aligned} \tag{139}$$

to the order of θ^2. If ϵ is a small quantity,

$$I_1 = \int_0^\epsilon \frac{(-\tau^2)^{\frac{1}{2}} a_0\theta\, d\theta}{-\mathrm{i}\theta^2\left[\frac{(2n-1)\pi}{16\tau^2} + \tau\cot\gamma\right]} + \int_\epsilon^\infty \frac{(\theta^2-\tau^2)^{\frac{1}{2}} \sin(a_0\theta)\, J_0(a\theta)\, d\theta}{l(\theta)}. \tag{140}$$

The first integral, being of the order θ^{-1}, diverges, indicating resonance when $\gamma = \dfrac{(2n-1)\pi}{2\tau}$, i.e.

$$k\delta = \frac{(2n-1)\pi}{2\tau} = \frac{(2n-1)\pi k}{2h},$$

$$\delta = \frac{(2n-1)\pi}{2h},$$

i.e.
$$p = \frac{(2n-1)\pi}{2\delta}\left(\frac{E(1-\nu)}{\rho(1+\nu)(1-2\nu)}\right)^{\frac{1}{2}}. \tag{141}$$

These resonant frequencies are equal to those of a rod of the elastic material fixed at one end, free at the other, and constrained at the sides so that all lateral movement vanishes there. It is easy to see that this result holds for any axisymmetrical vertical stress distribution which is finite at the centre. The stress distribution under a rigid plate falls into this category and thus a weightless rigid plate on the surface of the elastic medium will resonate at these frequencies.

A solution may be obtained for the case when all displacements vanish at the lower boundary, $z = 0$. This is so if the elastic medium is securely stuck to the base, preventing sliding, but it is to be expected that it will not differ very much from the previous case. Choose a solution with four arbitrary constants illustrated by

$$u = \left[\frac{A_1 x\,e^{-\alpha z}}{h^2} + \frac{A_2 x\,e^{\alpha z}}{h^2} - \frac{C_1 \beta x\,e^{-\beta z}}{k^2} + \frac{C_2 \beta x\,e^{\beta z}}{k^2}\right] J_1(xr), \tag{142}$$

$$w = \left[\frac{A_1 \alpha\,e^{-\alpha z}}{h^2} - \frac{A_2 \alpha\,e^{\alpha z}}{h^2} - \frac{C_1 x^2\,e^{-\beta z}}{k^2} - \frac{C_2 x^2\,e^{\beta z}}{k^2}\right] J_0(xr). \tag{143}$$

It can then be shown that the solution, when all displacements vanish at the lower boundary, is given by

$$w(r,\delta) = \frac{P\,e^{ipt}}{\mu\pi r_0}\int_0^\infty \frac{[x^2\cosh(\alpha\delta)\sinh(\beta\delta) - \alpha\beta\cosh(\beta\delta)\sinh(\alpha\delta)]\sin(xr_0)\,J_0(xr)}{\beta x^2 l_2(x)}\,dx + \text{free wave}, \tag{144}$$

$$l_2(x) = 2(x^2 - \tfrac{1}{2}k^2) - \frac{1}{x^2}[(x^2 - \tfrac{1}{2}k^2)^2 + x^4]\cosh(\alpha\delta)\cosh(\beta\delta) + [\alpha^2\beta^2 + (x^2 - \tfrac{1}{2}k^2)^2]\frac{\sinh(\alpha\delta)\sinh(\beta\delta)}{\alpha\beta}. \tag{145}$$

As $x \to 0$,

$$l_2(x) \doteqdot \frac{-k^4}{4x^2}\cosh(\alpha\delta)\cosh(\beta\delta), \tag{146}$$

and the integrand becomes

$$\frac{4\alpha\cosh(\beta\delta)\sinh(\alpha\delta)\,xr_0}{k^4\cosh(\alpha\delta)\cosh(\beta\delta)}. \tag{147}$$

Choose $\coth[\gamma(-\tau^2)^{\frac{1}{2}}] = 0$, as before, and the integrand is of the order x^{-1} and the integral diverges. It follows that the same resonant frequencies occur whether the stratum is free to slide on the lower boundary or is completely restrained there.

(*b*) *Static displacements*

It has been intimated that the static vertical displacement of a rigid plate on a stratum involves two intractable integral equations. However, close bounds may be obtained to the displacement by using the principle evolved earlier. Take the limit of expression (125) as $h, k \to 0$, and we have the static displacements due to the stress distribution of the static semi-infinite case. It will be shown, and is expected physically, that this stress distribution is greater at the outside and smaller at the centre than that caused by a rigid plate on the stratum. This stress distribution applied to the surface of a stratum gives displacements of the shape shown in figure 9. Take the limit as $h, k \to 0$ of expression (125) and it is found that

$$w(0,\delta) = \frac{P(\lambda+2\mu)}{4\pi\mu r_0(\lambda+\mu)}\int_0^\infty \frac{\sinh^2 x\sin(x/R)\,dx}{R(x+\cosh x\sinh x)(x/R)}, \tag{148}$$

$$w(r_0,\delta) = \frac{P(\lambda+2\mu)}{4\pi\mu r_0(\lambda+\mu)}\int_0^\infty \frac{\sinh^2 x\sin(x/R)\,J_0(x/R)\,dx}{R(x+\cosh x\sinh x)(x/R)}, \tag{149}$$

$$w_a(\delta) = \frac{P(\lambda+2\mu)}{4\pi\mu r_0(\lambda+\mu)}\int_0^\infty \frac{\sinh^2 x\sin^2(x/R)\,dx}{R(x+\cosh x\sinh x)(x/R)^2}, \tag{150}$$

where

$$R = \delta/r_0 \quad\text{and}\quad w_a(\delta) = \text{'average' displacement.} \tag{151}$$

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There appears no simple way of evaluating these integrals because of the difficulty of determining the complex poles of the integrand. Write

$$w(0,\delta) = \frac{P(\lambda+2\mu)\,K_1}{8\mu r_0(\lambda+\mu)}, \quad K_1 = \frac{2}{\pi R}\int_0^\infty \frac{\sinh^2 x \sin(x/R)\,\mathrm{d}x}{(x+\cosh x \sinh x)\,(x/R)}, \tag{152}$$

$$w(r_0,\delta) = \frac{P(\lambda+2\mu)\,K_2}{8\mu r_0(\lambda+\mu)}, \quad K_2 = \frac{2}{\pi R}\int_0^\infty \frac{\sinh^2(x)\sin(x/R)\,J_0(x/R)\,\mathrm{d}x}{(x+\cosh x \sinh x)\,(x/R)}, \tag{153}$$

$$w_a(\delta) = \frac{P(\lambda+2\mu)\,K_3}{8\mu r_0(\lambda+\mu)}, \quad K_3 = \frac{2}{\pi R}\int_0^\infty \frac{\sinh^2 x \sin^2(x/R)\,\mathrm{d}x}{(x+\cosh x \sinh x)\,(x/R)^2}, \tag{154}$$

$$K_3 \doteqdot \frac{2}{\pi R}\left[\int_0^\Phi \frac{\sinh^2 x \sin^2(x/R)\,\mathrm{d}x}{(x+\cosh x \sinh x)\,(x/R)^2} + \int_\Phi^\infty \frac{\sin^2(x/R)\,\mathrm{d}x}{(x/R)^2}\right], \tag{155}$$

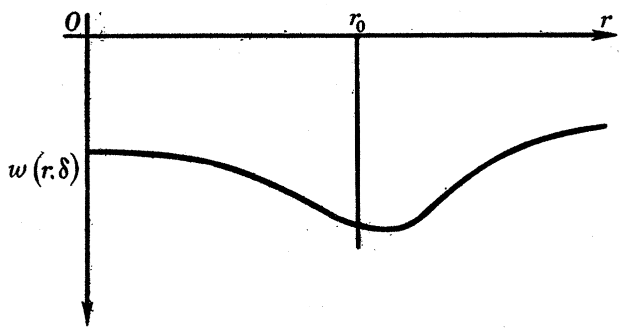

FIGURE 9

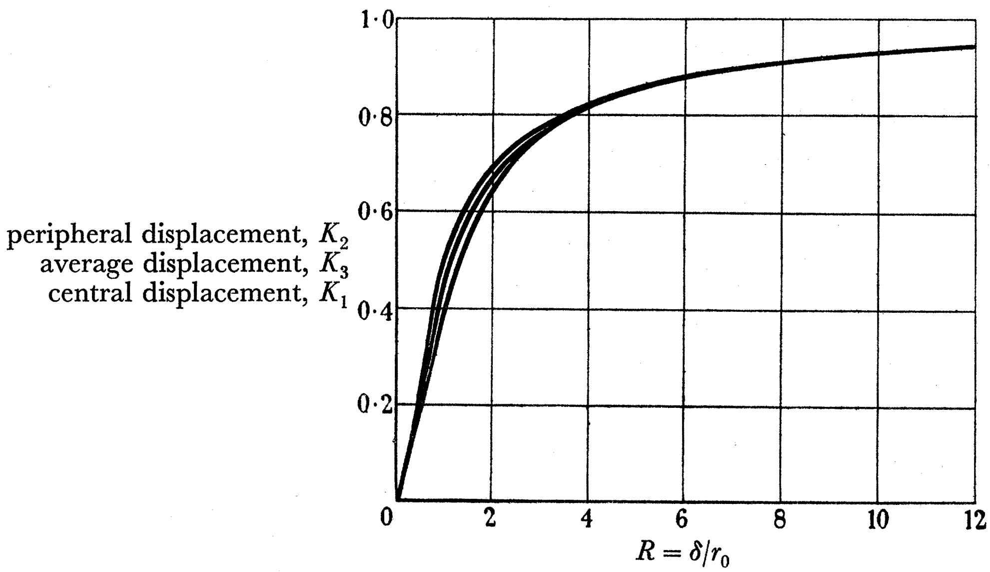

FIGURE 10. Elastic stratum stiffness factors, vertical translation.

where Φ is a number large enough to make $\dfrac{\sinh^2 x}{(x+\cosh x \sinh x)} \doteqdot 1$.

$$K_3 = \frac{2}{\pi R}\left[\int_0^\infty \frac{\sin^2(x/R)\,\mathrm{d}x}{(x/R)^2} + \int_0^\Phi \frac{(\sinh^2 x - x - \cosh x \sinh x)\sin^2(x/R)\,\mathrm{d}x}{(x+\cosh x \sinh x)\,(x/R)^2}\right], \tag{156}$$

$$\int_0^\infty \frac{\sin^2(x/R)\,\mathrm{d}x}{(x/R)^2} = \frac{\pi R}{2}, \tag{157}$$

and then

$$K_3 = 1 - \int_0^\Phi \frac{2(x+\cosh x \sinh x - \sinh^2 x)\sin^2(x/R)\,\mathrm{d}x}{\pi R(x+\cosh x \sinh x)\,(x/R)^2}. \tag{158}$$

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This latter integral was evaluated numerically. The factors K_1, K_2 are treated similarly and the three factors are shown in figure 10. Remembering that the slope of the surface is opposite that of the vertical dynamic case, it follows that $w(0,\delta)$ is the highest point, and using the reciprocal theorem, as before, it follows that K for a rigid plate lies between K_1 and K_3. The evaluated results show that, as long as the stratum is not too shallow, K_1 and K_3 are close together and we have reasonable bounds for K. It is to be expected that K will be much closer to K_3 than to K_1, and experiment indicates that K_3 is a good approximation to K even for very shallow strata. As R ranges from 0 to ∞ these factors range from 0 to 1. When $R=\infty$, $K=1$, the result is that of Boussinesq for a semi-infinite elastic space.

4. Rotation about a horizontal axis

Semi-infinite elastic space

The solution to the case when the rigid circular plate attached to the surface of a semi-infinite elastic space is rotated about a horizontal axis by a couple follows from the sum of the solutions given by equations (8) to (10) and (14) to (16) with $m=1$:

$$u=\left[\frac{-A(x)\,\mathrm{e}^{-\alpha z}}{h^2}+\frac{C(x)\,\beta\,\mathrm{e}^{-\beta z}}{k^2}\right]\frac{\partial J_1(xr)}{\partial r}\begin{matrix}\cos\\ \sin\end{matrix}\,\theta\,\mathrm{e}^{\mathrm{i}pt}, \tag{159}$$

$$v=\left[\frac{A(x)\,\mathrm{e}^{-\alpha z}}{h^2}-\frac{C(x)\,\beta\,\mathrm{e}^{-\beta z}}{k^2}\right]\frac{J_1(xr)}{r}\begin{matrix}\sin\\ -\cos\end{matrix}\,\theta\,\mathrm{e}^{\mathrm{i}pt}, \tag{160}$$

$$w=\left[\frac{A(x)\,\alpha\,\mathrm{e}^{-\alpha z}}{h^2}-\frac{C(x)\,x^2\,\mathrm{e}^{-\beta z}}{k^2}\right]J_1(xr)\begin{matrix}\cos\\ \sin\end{matrix}\,\theta\,\mathrm{e}^{\mathrm{i}pt}. \tag{161}$$

(*a*) *Static solution*

Expand above solutions in terms of h and k to the first order, put $(-A/h^2+C/k^2)=A_1$, $(C-A)=C_1$ and take the limit as $h,k\to 0$ and the static solutions follow as

$$u=\left[A_1(x)+\frac{C_1(x)\,z}{2x}\right]\mathrm{e}^{-xz}\frac{\partial J_1(xr)}{\partial r}\begin{matrix}\cos\\ \sin\end{matrix}\,\theta, \tag{162}$$

$$v=\left[-A_1(x)-\frac{C_1(x)\,z}{2x}\right]\mathrm{e}^{-xz}\frac{J_1(xr)}{r}\begin{matrix}\sin\\ -\cos\end{matrix}\,\theta, \tag{163}$$

$$w=\left[-A_1(x)\,x-\frac{NC_1(x)}{2x}-\frac{C_1(x)\,z}{2}\right]\mathrm{e}^{-xz}J_1(xr)\begin{matrix}\cos\\ \sin\end{matrix}\,\theta, \tag{164}$$

$$\widehat{zz}=\left[\frac{\lambda(N-1)\,C_1(x)}{2}+2\mu\left\{A_1(x)\,x^2+\frac{(N-1)\,C_1(x)}{2}+\frac{C_1(x)\,xz}{2}\right\}\right]\mathrm{e}^{-xz}J_1(xr)\begin{matrix}\cos\\ \sin\end{matrix}\,\theta, \tag{165}$$

$$\widehat{rz}=\mu\left[-2xA_1(x)+\frac{C_1(x)}{2x}-\frac{zC_1(x)}{2}-\frac{NC_1(x)}{2x}-\frac{zC_1(x)}{2}\right]\mathrm{e}^{-xz}\frac{\partial J_1(xr)}{\partial r}\begin{matrix}\cos\\ \sin\end{matrix}\,\theta, \tag{166}$$

$$\widehat{z\theta}=\mu\left[A_1(x)\,x+\frac{NC_1(x)}{2x}+\frac{C_1(x)\,z}{2}+xA_1(x)+\frac{zC_1(x)}{2}-\frac{C_1(x)}{2x}\right]\mathrm{e}^{-xz}\frac{J_1(xr)}{r}\begin{matrix}\sin\\ -\cos\end{matrix}\,\theta, \tag{167}$$

where

$$N=\frac{k^2+h^2}{k^2-h^2}=\frac{(\lambda+3\mu)}{(\lambda+\mu)}. \tag{168}$$

On the top surface, $z=0$, the shear stress is to be zero everywhere, i.e.

$$\widehat{rz} = \widehat{z\theta} = 0 \quad (z=0),$$

i.e.
$$C_1 = \frac{-4A_1x^2}{(N-1)},$$

and, with this value of $C_1(x)$,

$$w(r,\theta,0) = A_1(x)\,x\frac{(N+1)}{(N-1)}J_1(xr)\,{\cos \atop \sin}\,\theta, \tag{169}$$

$$\widehat{zz}(r,\theta,0) = -2(\lambda+\mu)\,A_1(x)\,x^2J_1(xr)\,{\cos \atop \sin}\,\theta. \tag{170}$$

Change $A_1(x)$ to $A_2(x)$ and generalize the solution by integrating from 0 to ∞ with respect to x:

$$w(r,\theta,0) = \int_0^\infty A_2(x)\,xJ_1(xr)\,\mathrm{d}x\,{\cos \atop \sin}\,\theta, \tag{171}$$

$$\widehat{zz}(r,\theta,0) = \frac{-2\mu(\lambda+\mu)}{(\lambda+2\mu)}\int_0^\infty A_2(x)\,x^2J_1(xr)\,\mathrm{d}x\,{\cos \atop \sin}\,\theta. \tag{172}$$

If ϕ is the angle of rotation of the circular rigid plate about a horizontal axis the following boundary conditions are still to be satisfied:

$$\left.\begin{aligned} w(r,\theta,0) &= \phi r\sin\theta \quad (r\leqslant r_0),\\ \widehat{zz}(r,\theta,0) &= 0 \qquad\qquad (r>r_0).\end{aligned}\right\} \tag{173}$$

The angle θ in the horizontal plane is measured from the horizontal axis of rotation. These last two boundary conditions are satisfied if $A_2(x)$ can be found to satisfy the following dual integral equations:

$$\left.\begin{aligned} \int_0^\infty A_2(x)\,xJ_1(xr)\,\mathrm{d}x &= \phi r \quad (r\leqslant r_0),\\ \int_0^\infty A_2(x)\,x^2J_1(xr)\,\mathrm{d}x &= 0 \quad (r>r_0).\end{aligned}\right\} \tag{174}$$

These two equations are recognized as a special case of the two dual integral equations of Busbridge, equations (21) to (23). Noting that

$$\Gamma(\tfrac{1}{2}) = \pi^{\frac{1}{2}} \quad \text{and} \quad J_{\frac{3}{2}}(x) = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}}\left(\frac{\sin x}{x}-\cos x\right)$$

it follows that
$$A_2(x) = \frac{4\phi r_0}{\pi x^2}\left[\frac{\sin(xr_0)}{xr_0}-\cos(xr_0)\right]. \tag{175}$$

This value for $A_2(x)$ gives

$$\widehat{zz}(r,\theta,0) = \frac{-2\mu(\lambda+\mu)\,4\phi r_0\sin\theta}{(\lambda+2\mu)\,\pi}\int_0^\infty\left[\frac{\sin(xr_0)}{xr_0}-\cos(xr_0)\right]J_1(xr)\,\mathrm{d}x. \tag{176}$$

It follows (Watson 1944, p. 405) that

$$\widehat{zz}(r,\theta,0) = \frac{-2\mu(\lambda+\mu)\,4\phi r\sin\theta}{(\lambda+2\mu)\,\pi(r_0^2-r^2)^{\frac{1}{2}}}. \tag{177}$$

This stress distribution is to be expected as it is a superposition of a linear increase with respect to r on the stress distribution of the vertical translation case.

If M is the turning moment necessary to rotate the plate through an angle ϕ about the horizontal axis, then

$$M = \frac{-2\mu(\lambda+\mu)\,4\phi}{(\lambda+2\mu)\,\pi}\int_0^{2\pi}\sin^2\theta\,\mathrm{d}\theta\int_0^{r_0}\frac{r^3\,\mathrm{d}r}{(r_0^2-r^2)^{\frac{1}{2}}}, \tag{178}$$

$$M = \frac{16\phi\mu(\lambda+\mu)\,r_0^3}{3(\lambda+2\mu)},$$

$$\phi = \frac{3M}{16\mu r_0^3(1-\tau^2)}. \tag{179}$$

This simple expression should be of interest in discussing the rocking of towers on circular bases. It could be used as a criterion of stability of a tower by equating the righting moment, as given above, to the overturning moment caused by the weight of the tower.

(*b*) *Dynamic solution*

With $m = 1$ in the solutions, equations (8) to (10) and (14) to (16), the following are obtained:

$$u = \left[\frac{-A(x)\,\mathrm{e}^{-\alpha z}}{h^2}+\frac{C(x)\,\beta\,\mathrm{e}^{-\beta z}}{k^2}\right]\frac{\partial J_1(xr)}{\partial r}\,\frac{\cos}{\sin}\,\theta\,\mathrm{e}^{\mathrm{i}pt}, \tag{180}$$

$$v = \left[\frac{A(x)\,\mathrm{e}^{-\alpha z}}{h^2}-\frac{C(x)\,\beta\,\mathrm{e}^{-\beta z}}{k^2}\right]\frac{J_1(xr)}{r}\,\frac{\sin}{-\cos}\,\theta\,\mathrm{e}^{\mathrm{i}pt}, \tag{181}$$

$$w = \left[\frac{A(x)\,\alpha\,\mathrm{e}^{-\alpha z}}{h^2}-\frac{C(x)\,x^2\,\mathrm{e}^{-\beta z}}{k^2}\right]J_1(xr)\,\frac{\cos}{\sin}\,\theta\,\mathrm{e}^{\mathrm{i}pt}, \tag{182}$$

$$\widehat{zz} = \mu\left[A(x)\left(\frac{\lambda}{\mu}-\frac{2\alpha^2}{h^2}\right)\mathrm{e}^{-\alpha z}+\frac{C(x)\,\beta 2x^2\,\mathrm{e}^{-\beta z}}{k^2}\right]J_1(xr)\,\frac{\cos}{\sin}\,\theta\,\mathrm{e}^{\mathrm{i}pt}, \tag{183}$$

$$\widehat{zr} = \mu\left[\frac{A(x)\,2\alpha\,\mathrm{e}^{-\alpha z}}{h^2}-C(x)\,\beta\left(\frac{\beta}{k^2}+\frac{x^2}{\beta k^2}\right)\mathrm{e}^{-\beta z}\right]\frac{\partial J_1(xr)}{\partial r}\,\frac{\cos}{\sin}\,\theta\,\mathrm{e}^{\mathrm{i}pt}, \tag{184}$$

$$\widehat{r\theta} = \mu\left[\frac{-A(x)\,2\alpha\,\mathrm{e}^{-\alpha z}}{h^2}+C(x)\,\beta\left(\frac{\beta}{k^2}+\frac{x^2}{\beta k^2}\right)\mathrm{e}^{-\beta z}\right]\frac{J_1(xr)}{r}\,\frac{\sin}{-\cos}\,\theta\,\mathrm{e}^{\mathrm{i}pt}. \tag{185}$$

Because the same function of x appears in both $\widehat{zr}$ and $\widehat{r\theta}$, it is possible to choose $C(x)$ in terms of $A(x)$ so that both $\widehat{zr} = 0$, $\widehat{r\theta} = 0$, $z = 0$.

As in the vertical translation case $A(x)$ is chosen to satisfy the normal static stress on the free surface $z = 0$. Take this stress distribution as the static stress distribution, just evaluated, i.e.

$$\widehat{zz} = \frac{Kr\sin\theta}{(r_0^2-r^2)^{\frac{1}{2}}}. \tag{186}$$

Express $\widehat{zz}$ by the Fourier–Bessel theorem, and

$$\widehat{zz} = \int_0^\infty J_1(xr)\,x\,\mathrm{d}x\left[\int_0^{r_0}\frac{K\rho\sin\theta J_1(x\rho)\,\rho\,\mathrm{d}\rho}{(r_0^2-\rho^2)^{\frac{1}{2}}}\right], \tag{187}$$

$$I = \int_0^{r_0}\frac{\rho^2 J_1(x\rho)\,\mathrm{d}\rho}{(r_0^2-\rho^2)^{\frac{1}{2}}} = \int_0^{\frac{1}{2}\pi} r_0^2 J_1(xr_0\sin\psi)\sin^2\psi\,\mathrm{d}\psi. \tag{188}$$

This integral is now the special case of Sonine's first integral, equation (24), where $z = (xr_0)$, $\mu = 1$, $\nu = -\frac{1}{2}$. Using $\Gamma(\frac{1}{2}) = \pi^{\frac{1}{2}}$ and $J_{\frac{3}{2}}(y) = \left(\frac{2}{\pi y}\right)^{\frac{1}{2}}\left(\frac{\sin y}{y}-\cos y\right)$, it follows that

$$I = \frac{r_0}{x}\left[\frac{\sin(xr_0)}{xr_0}-\cos(xr_0)\right]. \tag{189}$$

The displacement $w(r,\theta,0)$ follows directly and is examined by contour integration and a free wave added to make the whole a travelling wave at infinity. The displacement is

$$w(r,\theta,0)=\frac{Kr_0\sin\theta\,\mathrm{e}^{\mathrm{i}pt}}{4\mu}\int_0^\infty\frac{\alpha k^2[\sin(xr_0)-xr_0\cos(xr_0)]\,J_1(xr)\,\mathrm{d}x}{xr_0f(x)}$$
$$-\frac{\mathrm{i}\pi Kr_0\sin\theta\,\mathrm{e}^{\mathrm{i}pt}\alpha_1k^2[\sin(x_1r_0)-x_1r_0\cos(x_1r_0)]\,J_1(x_1r_0)}{4\mu x_1r_0f'(x_1)}.\qquad(190)$$

Change the variable x to θ and 'average' the displacement over $r\leqslant r_0$ in the same manner as before and then

$$\phi_a=\frac{9M}{16\mu\pi r_0^3}\int_0^\infty\frac{(\theta^2-\tau^2)^{\frac{1}{2}}[\sin(a_0\theta)-a_0\theta\cos(a_0\theta)]^2\,\mathrm{d}\theta}{(a_0\theta)^3f(\theta)}$$
$$-\frac{\mathrm{i}9M(\theta_1^2-\tau^2)^{\frac{1}{2}}[\sin(a_0\theta_1)-a_0\theta_1\cos(a_0\theta_1)]^2}{16\mu r_0^3f'(\theta_1)\,(a_0\theta_1)^3}$$
$$=\frac{M}{\mu r_0^3}(f_{1a}+\mathrm{i}f_{2a}).\qquad(191)$$

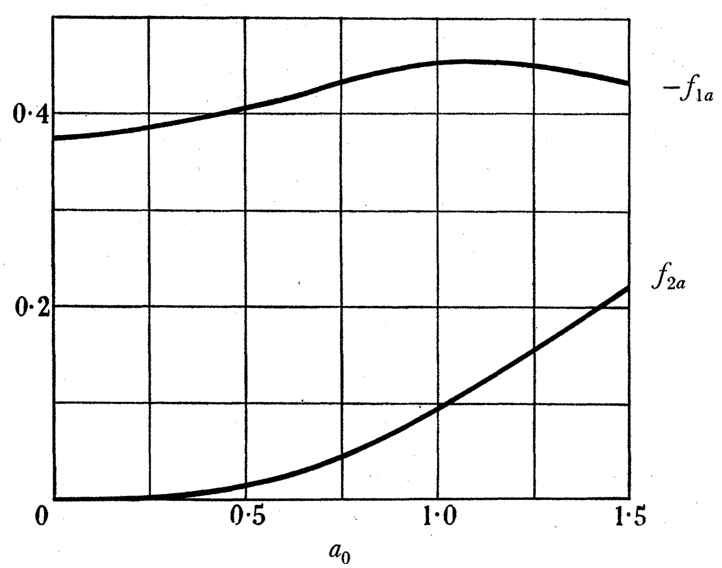

FIGURE 11. Semi-infinite space, rotation in vertical plane; $\tau^2=\frac{1}{2}$.

The Cauchy principal value of this integral is implied by the contour integration. ϕ_p follows by putting $\frac{2}{3}J_1(a_0\theta)$ in place of $[\sin(a_0\theta)-a_0\theta\cos(a_0\theta)]/(a_0\theta)^2$ in ϕ_a. If ϕ is the true angle of rotation of a rigid plate then, as before, $\phi_p<\phi<\phi_a$. Only ϕ_a has been evaluated.

Evaluation follows exactly as before. Write

$$I=\int_0^\infty\frac{(\theta^2-\tau^2)^{\frac{1}{2}}[\sin(a_0\theta)-a_0\theta\cos(a_0\theta)]^2\,\mathrm{d}\theta}{f(\theta)\,(a_0\theta)^3}$$
$$=\int_0^\infty\frac{(\theta^2-\tau^2)^{\frac{1}{2}}[\sin(a_0\theta)-a_0\theta\cos(a_0\theta)]\sin(a_0\theta)\,\mathrm{d}\theta}{f(\theta)\,(a_0\theta)^3}$$
$$-\int_0^\infty\frac{(\theta^2-\tau^2)^{\frac{1}{2}}[\sin(a_0\theta)-a_0\theta\cos(a_0\theta)]\cos(a_0\theta)\,\mathrm{d}\theta}{f(\theta)\,(a_0\theta)^2}.\qquad(192)$$

These two integrals are evaluated by replacing the single terms $\sin(a_0\theta)$, $\cos(a_0\theta)$ by $\mathrm{e}^{\mathrm{i}a_0\theta}$ and integrating these new integrals around the previous contour separating out the required integrals as before. The evaluated results are shown in figure 11.

Elastic stratum

If the plate is attached to the free surface of an elastic stratum and rotated the displacements follow as before and

$$w(r,\theta,0) = \frac{Kr_0 \sin\theta\, \mathrm{e}^{\mathrm{i}pt}}{4\mu}\int_0^\infty \frac{\alpha k^2 \sinh(\alpha\delta)\sinh(\beta\delta)\,[\sin(xr_0)-xr_0\cos(xr_0)]\,J_1(xr)\,\mathrm{d}x}{l_1(x)\,xr_0} + \text{free wave}. \tag{193}$$

It is interesting to note that resonance does not exist in this case. An analysis, as carried out for the vertical case, will reveal that for any value of δ the integrand close to $\theta = 0$ is of the order θ and thus does not diverge. If the integrals are expressed with respect to θ rather than x it is seen that the frequency only occurs in factors such as $[\sin(a_0\theta)-a_0\theta\cos(a_0\theta)]/a_0\theta$, and it is these factors that substantially determine the frequency response.

5. Rotation about a vertical axis

Rotation of the plate about a vertical axis is the simplest of the four modes of oscillation because no dilatation in the medium exists. The plate on a semi-infinite space has been solved exactly by Reissner & Sagoci using a system of oblate spheroidal co-ordinates. It is solved again here by the approximate methods of this paper, for comparison. The plate on an elastic stratum, mentioned but not solved by Reissner, is considered also.

Semi-infinite elastic space

The solution of the wave equation given by equations (11) to (13) when $m = 0$ is

$$\left.\begin{aligned} u_2 &= 0,\\ v_2 &= \frac{-B\,J_1(xr)}{x}\,\mathrm{e}^{-\beta z}\,\mathrm{e}^{\mathrm{i}pt},\\ w_2 &= 0. \end{aligned}\right\} \tag{194}$$

This solution makes all surface stresses zero except the shear stress $\widehat{z\theta}$.

$$\widehat{z\theta} = \frac{\mu B\beta\,J_1(xr)}{x}\,\mathrm{e}^{-\beta z}\,\mathrm{e}^{\mathrm{i}pt}. \tag{195}$$

The only boundary condition still to satisfy is that the stress $\widehat{z\theta}$ over $0 \leqslant r \leqslant r_0$ is that of the static case. By using the two dual integral equations, as before, it can be easily shown that if M is the static twisting moment applied to the plate then the stress distribution is

$$\widehat{z\theta} = \frac{3Mr}{4\pi r_0^3(r_0^2-r^2)^{\frac{1}{2}}} \quad (r \leqslant r_0); \qquad \widehat{z\theta} = 0 \quad (r > r_0). \tag{196}$$

It is to be noticed that the factor $r(r_0^2-r^2)^{-\frac{1}{2}}$ also occurs in the case of rotation about a horizontal axis, and the two rotation cases have much in common. The displacement caused by this stress distribution is

$$v(r,0) = \frac{3M\,\mathrm{e}^{\mathrm{i}pt}}{4\mu\pi r_0^2}\int_0^\infty \frac{[\sin(xr_0)-xr_0\cos(xr_0)]\,J_1(xr)\,\mathrm{d}x}{\beta x r_0}. \tag{197}$$

There are no free waves to be added because free waves cannot exist. In equation (195) put $\widehat{z\theta} = 0$, indicating no surface stress, and it follows that $B(x) = 0$, i.e. $v_2 = 0$. This is further emphasized by the resulting integral for $v(r, 0)$. The residue at the singularity $x = \pm k$, i.e. $\beta = 0$, is zero and no doubt arises as to the value of the integral.

'Average' the displacement as in the vertical rotation case, and the 'average' angle of rotation ϕ_a becomes

$$\phi_a = \frac{9M\,e^{ipt}}{8\mu\pi r_0^3}\int_0^\infty \frac{[\sin(a_0\theta)-a_0\theta\cos(a_0\theta)]^2\,d\theta}{(a_0\theta)^3\,(\theta^2-1)^{\frac{1}{2}}}. \tag{198}$$

The angle corresponding to the displacement at the periphery is given by

$$\phi_p = \frac{3M\,e^{ipt}}{4\mu\pi r_0^3}\int_0^\infty \frac{[\sin(a_0\theta)-(a_0\theta)\cos(a_0\theta)]\,J_1(a_0\theta)\,d\theta}{a_0\theta(\theta^2-1)^{\frac{1}{2}}}. \tag{199}$$

It follows that the angle ϕ of a rigid plate is such that $\phi_p<\phi<\phi_a$. The integrals have an imaginary part, representing energy lost to infinity in the form of body waves shown in the integral by $\int_0^1$, and a real part $\int_1^\infty$. The angles are now expressed as

$$\phi_a = \frac{M\,e^{ipt}}{\mu r_0^3}(f_{1a}+if_{2a}), \tag{200}$$

$$\phi_p = \frac{M\,e^{ipt}}{\mu r_0^3}(f_{1p}+if_{2p}). \tag{201}$$

It is necessary to evaluate

$$\begin{aligned} I &= \int_0^\infty \frac{[\sin(a_0\theta)-a_0\theta\cos(a_0\theta)]^2\,d\theta}{(\theta^2-1)^{\frac{1}{2}}\,(a_0\theta)^3} \\ &= \int_0^\infty \frac{[\sin(a_0\theta)-a_0\theta\cos(a_0\theta)]\sin(a_0\theta)\,d\theta}{(\theta^2-1)^{\frac{1}{2}}\,(a_0\theta)^3} - \int_0^\infty \frac{[\sin(a_0\theta)-a_0\theta\cos(a_0\theta)]\cos(a_0\theta)\,d\theta}{(\theta^2-1)^{\frac{1}{2}}\,(a_0\theta)^2} \\ &= I_1 - I_2. \end{aligned} \tag{202}$$

These integrals are treated separately. For I_1, integrate J_1 around the general contour changing the sign of β at $x=-1$:

$$J_1 = \int_{-\infty}^{\infty} \frac{[\sin(a_0\theta)-a_0\theta\cos(a_0\theta)]\,e^{ia_0\theta}\,d\theta}{(\theta^2-1)^{\frac{1}{2}}\,(a_0\theta)^3}. \tag{203}$$

The integral around the semicircle vanishes and the residues at ± 1 are zero, i.e.

$$\lim_{\theta\to 1} \frac{(\theta-1)\,[\sin(a_0\theta)-a_0\theta\cos(a_0\theta)]\,e^{ia_0\theta}}{(\theta+1)^{\frac{1}{2}}\,(\theta-1)^{\frac{1}{2}}\,(a_0\theta)^3} = 0. \tag{204}$$

This contour integration yields

$$I_1 = -i\int_0^1 \frac{[\sin(a_0\theta)-a_0\theta\cos(a_0\theta)]\sin(a_0\theta)\,d\theta}{(1-\theta^2)^{\frac{1}{2}}\,(a_0\theta)^3} + \int_0^1 \frac{[\sin(a_0\theta)-a_0\theta\cos(a_0\theta)]\cos(a_0\theta)\,d\theta}{(1-\theta^2)^{\frac{1}{2}}\,(a_0\theta)^3}. \tag{205}$$

Similarly,

$$I_2 = -i\int_0^1 \frac{[\sin(a_0\theta)-a_0\theta\cos(a_0\theta)]\cos(a_0\theta)\,d\theta}{(1-\theta^2)^{\frac{1}{2}}\,(a_0\theta)^2} - \int_0^1 \frac{[\sin(a_0\theta)-a_0\theta\cos(a_0\theta)]\sin(a_0\theta)\,d\theta}{(1-\theta^2)^{\frac{1}{2}}\,(a_0\theta)^2}. \tag{206}$$

These are integrated by expanding the terms containing $(a_0\theta)$ as a power series and then using the substitution $\theta = \sin\psi$.

Finally,

$$I = (0\cdot524+0\cdot105a_0^2-0\cdot0332a_0^4+0\cdot004a_0^6-0\cdot0002a_0^8+\ldots)$$
$$-\mathrm{i}(0\cdot074a_0^3-0\cdot0118a_0^5+0\cdot0009a_0^7-0\cdot00002a_0^9+\ldots), \quad (207)$$

and if we write
$$\phi_a = \frac{M\,\mathrm{e}^{\mathrm{i}pt}}{\mu r_0^3}(f_{1a}+\mathrm{i}f_{2a}), \quad (208)$$

then
$$\left.\begin{aligned} f_{1a} &= -0\cdot187-0\cdot0376a_0^2+0\cdot0119a_0^4-0\cdot0013a_0^6+0\cdot00006a_0^8,\\ f_{2a} &= 0\cdot0264a_0^3-0\cdot0042a_0^5+0\cdot00031a_0^7-0\cdot00001a_0^9.\end{aligned}\right\} \quad (209)$$

The integral for ϕ_p may be evaluated similarly and the computed values are shown in figure 12. These two bounds for ϕ_a, ϕ_p are close together up to $a_0 = 2$ and are seen to enclose the true ϕ as calculated by Reissner & Sagoci. The variation of these functions with a_0 is seen to be very similar to the case of rotation about a horizontal axis, this frequency variation being substantially effected by the common term

$$[\sin(a_0\theta)-a_0\theta\cos(a_0\theta)]^2/(a_0\theta)^2.$$

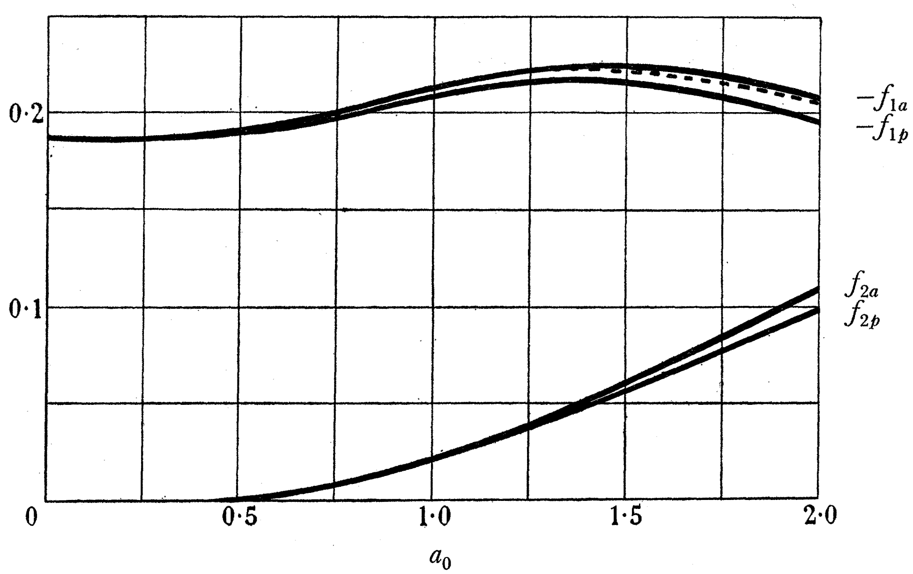

FIGURE 12. Semi-infinite space, rotation in horizontal plane. ---- Sagoci's exact solution.

Elastic stratum

It is convenient to write $\beta_1 = (k^2-x^2)^{\frac{1}{2}} = \mathrm{i}\beta$ and to obtain as a solution from equations (11) to (13)

$$\left.\begin{aligned} u_2 &= 0,\\ v_2 &= [A(x)\cos(\beta_1 z)+B(x)\sin(\beta_1 z)]\,J_1(xr)\,\mathrm{e}^{\mathrm{i}pt},\\ w_2 &= 0.\end{aligned}\right\} \quad (210)$$

Such a solution already satisfies the condition that all surface stresses are zero except $\widehat{z\theta}$. If $z = 0$ is chosen as the upper free surface and $z = \delta$ is secured to a lower rigid stratum the following lower boundary condition is to be satisfied $v = 0$, $z = \delta$, i.e.

$$A(x)\cos(\beta_1\delta)+B(x)\sin(\beta_1\delta) = 0, \quad (211)$$

and then
$$v(r,z) = A(x)\,[\cos(\beta_1 z)-\cot(\beta_1\delta)\sin(\beta_1 z)]\,J_1(xr)\,\mathrm{e}^{\mathrm{i}pt}, \quad (212)$$
$$\widehat{z\theta}(r,z) = \mu A(x)\,[-\beta_1\sin(\beta_1 z)-\beta_1\cot(\beta_1\delta)\cos(\beta_1 z)]\,J_1(xr)\,\mathrm{e}^{\mathrm{i}pt}. \quad (213)$$

Generalize these solutions, and

$$v(r,0) = e^{ipt}\int_0^\infty A(x)\,J_1(xr)\,dx, \tag{214}$$

$$\widehat{z\theta}(r,0) = e^{ipt}\int_0^\infty \mu A(x)\,\beta_1\cot(\beta_1\delta)\,J_1(xr)\,dx. \tag{215}$$

Apply the static semi-infinite space stress distribution, i.e.

$$\widehat{z\theta}(r,0) = \frac{Kr}{(r_0^2-r^2)^{\frac{1}{2}}}. \tag{216}$$

Expansion by the Fourier–Bessel theorem, i.e.

$$\widehat{z\theta}(r,0) = \int_0^\infty J_1(xr)\,x\left[\int_0^{r_0}\frac{K\rho J_1(x\rho)\,\rho\,d\rho}{(r_0^2-\rho^2)^{\frac{1}{2}}}\right]dx$$

$$= \int_0^\infty \frac{Kr_0[\sin(xr_0)-xr_0\cos(xr_0)]\,J_1(xr)\,dx}{xr_0} \tag{217}$$

determines $A(x)$, and the angles of rotation become

$$\phi_a = \frac{9M\,e^{ipt}}{8\mu\pi r_0^3}\int_0^\infty \frac{[\sin(a_0\theta)-a_0\theta\cos(a_0\theta)]^2\sin[a_0R(1-\theta^2)^{\frac{1}{2}}]\,d\theta}{(a_0\theta)^3(1-\theta^2)^{\frac{1}{2}}\cos[a_0R(1-\theta^2)^{\frac{1}{2}}]}, \tag{218}$$

$$\phi_p = \frac{3M\,e^{ipt}}{4\mu\pi r_0^3}\int_0^\infty \frac{[\sin(a_0\theta)-a_0\theta\cos(a_0\theta)]\,J_1(a_0\theta)\sin[a_0R(1-\theta^2)^{\frac{1}{2}}]\,d\theta}{a_0\theta(1-\theta^2)^{\frac{1}{2}}\cos[a_0R(1-\theta^2)^{\frac{1}{2}}]}, \tag{219}$$

$$v(r,0) = \frac{3M\,e^{ipt}}{4\mu\pi r_0^2}\int_0^\infty \frac{[\sin(xr_0)-xr_0\cos(xr_0)]\,J_1(xr)\sin(\beta_1\delta)\,dx}{xr_0\beta_1\cos(\beta_1\delta)}. \tag{220}$$

In order to specify bounds for the true angle ϕ of a rigid plate it is necessary to examine this case further than is proposed. There are two effects to be considered. First, the propagation of waves outwards tends to make angles at the centre greater and angles at the periphery less with increasing a_0. Secondly, because it has been assumed that the stress distribution is that of the semi-infinite static case, the effect of a decreasing stratum depth is to decrease angles at the centre and increase them at the periphery. Without an actual evaluation of the displacement v for various a_0, R, it is not possible to tell whether ϕ will lie between ϕ_a and ϕ_p or between ϕ_a and the angle at the centre. However, it is possible to say that over some certain stratum depth depending on a_0, $\phi_p<\phi<\phi_a$. However, the integrals arising in a more complete discussion of this problem are tedious. The integral arising in ϕ_p is the only one which permits of a straightforward evaluation and is the only one which has been considered. It illustrates qualitatively what happens and experiments show that quantitatively it is a very reasonable estimation.

Whereas in the semi-infinite case free waves cannot exist they may exist in the case of a stratum. Equate surface stresses to zero:

$$\left.\begin{aligned}\cos(\beta_1\delta) &= 0,\\ \cos[a_0R(1-\theta^2)^{\frac{1}{2}}] &= 0,\\ \theta_n &= \pm\left[1-\frac{(2n-1)^2\pi^2}{4a_0^2R^2}\right]^{\frac{1}{2}}.\end{aligned}\right\} \tag{221}$$

The free waves are then given by

$$v_0(r,0) = \sum_{n=1}^{N} A_n J_1(\theta_n a). \tag{222}$$

There will be a finite number N of these waves depending on a_0 and R.

As before, the solution must be examined in order to determine if the addition of free waves is necessary to make the displacements at large radius into travelling waves. This may be done at the same time as the integral is evaluated. In the expression

$$\begin{aligned} v(r,0) &= \frac{3M\,\mathrm{e}^{\mathrm{i}pt}}{4\mu\pi r_0^2}\int_0^{\infty}\frac{[\sin(a_0\theta)-a_0\theta\cos(a_0\theta)]\sin[a_0R(1-\theta^2)^{\frac{1}{2}}]\,J_1(a\theta)\,\mathrm{d}\theta}{a_0\theta(1-\theta^2)^{\frac{1}{2}}\cos[a_0R(1-\theta^2)^{\frac{1}{2}}]} \\ &= \frac{3M\,\mathrm{e}^{\mathrm{i}pt}I}{4\mu\pi r_0^2}, \end{aligned} \tag{223}$$

substitute $$J_1(a\theta) = \tfrac{1}{2}[H_1^{(1)}(a\theta)+H_1^{(2)}(a\theta)],$$

and $$I = \frac{1}{2}\int_{-\infty}^{\infty}\frac{[\sin(a_0\theta)-a_0\theta\cos(a_0\theta)]\sin[a_0R(1-\theta^2)^{\frac{1}{2}}]\,H_1^{(1)}(a\theta)\,\mathrm{d}\theta}{a_0\theta(1-\theta^2)^{\frac{1}{2}}\cos[a_0R(1-\theta^2)^{\frac{1}{2}}]}. \tag{224}$$

Integrate I around the previous contour, changing the sign of $(1-\theta^2)^{\frac{1}{2}}$ at $\theta=-1$. However, the integrand is even with respect to this radical, and the change of sign makes no difference. The integral around the semicircular part of the contour vanishes if $a \geqslant a_0$. If $H_1^{(2)}(a\theta)$ had been used in place of $H_1^{(1)}(a\theta)$ then it would be necessary to integrate around an infinite semicircle in the lower half-plane in order to secure vanishing of the integral on the circular part of the contour. The integrand has poles at real values of θ corresponding to the free waves and also at imaginary values of θ, i.e. when

$$\cos[a_0R(1-\theta^2)^{\frac{1}{2}}] = 0,$$

i.e. $$a_0\theta_n = \left[a_0^2-\frac{(2n-1)^2\pi^2}{4R^2}\right]^{\frac{1}{2}}. \tag{225}$$

When θ_n is imaginary only the positive sign is to be taken and when θ_n is real, both signs. Call the N real roots $\theta_{\pm n}$ and the imaginary roots $\theta_m = \mathrm{i}x_m$ and, if R refers to residue, the contour integration yields

$$I = \frac{2\pi\mathrm{i}}{2}\sum_{m=N+1}^{\infty} R_{\theta_m} + \frac{\pi\mathrm{i}}{2}\sum_{n=1}^{N} R_{\pm\theta_n}, \tag{226}$$

$$R_{\theta_m} = \frac{[\sin(a_0\theta_m)-a_0\theta_m\cos(a_0\theta_m)]\,H_1^{(1)}(a\theta_m)}{R(a_0\theta_m)^2}, \tag{227}$$

$$R_{\pm\theta_n} = \frac{[\sin(a_0\theta_n)-a_0\theta_n\cos(a_0\theta_n)]\,2\mathrm{i}Y_1(a\theta_n)}{R(a_0\theta_n)^2}. \tag{228}$$

Consider the R_{θ_m} terms. These contain the factor

$$H_1^{(1)}(a\theta_m) = H_1^{(1)}(a\mathrm{i}x_m) \doteqdot \left(\frac{2}{\pi a\mathrm{i}x_m}\right)^{\frac{1}{2}}\mathrm{e}^{\mathrm{i}(a\mathrm{i}x_m-3\pi/4)} \tag{229}$$

as a becomes large. These factors decrease exponentially with distance and rapidly become negligible compared to the $R_{\pm\theta_n}$ terms which decrease as $a^{-\frac{1}{2}}$.

The term $Y_1(a\theta_n)$ which occurs in $R_{\pm\theta_n}$ indicates a standing wave decreasing with respect to distance a as $a^{-\frac{1}{2}}$ and consequently becomes the preponderant term at large a. Convert this standing wave at infinity to a travelling wave by the addition of the free waves

$$v_0 = \sum_{n=1}^{N} A_n J_n(a\theta_n), \tag{230}$$

$$A_n = \frac{-\pi \mathrm{i}[\sin(a_0\theta_n) - a_0\theta_n\cos(a_0\theta_n)]}{R(a_0\theta_n)^2}. \tag{231}$$

The wave factor occurring in the addition of these two waves is

$$-\mathrm{i}[J_1(a\theta_n) - \mathrm{i}Y_1(a\theta_n)] = -\mathrm{i}H_1^{(2)}(a\theta_n),$$

and this is a wave travelling outwards. The final solution for the displacements when $r \geq r_0$ is

$$\begin{aligned} v(r,0) = \frac{3M\mathrm{e}^{\mathrm{i}pt}}{4\mu\pi r_0^2}\Bigg[\pi\mathrm{i}\sum_{m=N+1}^{\infty}&\frac{[\sin(a_0\theta_m) - a_0\theta_m\cos(a_0\theta_m)]\,H_1^{(1)}(a\theta_m)}{R(a_0\theta_m)^2} \\ &+\pi\mathrm{i}\sum_{n=1}^{N}\frac{[\sin(a_0\theta_n) - a_0\theta_n\cos(a_0\theta_n)]\,\mathrm{i}Y_1(a\theta_n)}{R(a_0\theta_n)^2} \\ &-\pi\mathrm{i}\sum_{n=1}^{N}\frac{[\sin(a_0\theta_n) - a_0\theta_n\cos(a_0\theta_n)]\,J_1(a\theta_n)}{R(a_0\theta_n)^2}\Bigg], \end{aligned} \tag{232}$$

$$\phi_p = \frac{v(r_0,0)}{r_0} = \frac{M\mathrm{e}^{\mathrm{i}pt}}{\mu r_0^3}[f_{1p} + \mathrm{i}f_{2p}], \tag{233}$$

$$\begin{aligned} f_{1p} = \frac{3}{4\pi}\Bigg[\pi\mathrm{i}\sum_{m=N+1}^{\infty}&\frac{[\sin(a_0\theta_m) - a_0\theta_m\cos(a_0\theta_m)]\,H_1^{(1)}(a_0\theta_m)}{R(a_0\theta_m)^2} \\ &+\pi\mathrm{i}\sum_{n=1}^{N}\frac{[\sin(a_0\theta_n) - a_0\theta_n\cos(a_0\theta_n)]\,\mathrm{i}Y_1(a_0\theta_n)}{R(a_0\theta_n)^2}\Bigg], \end{aligned} \tag{234}$$

$$\mathrm{i}f_{2p} = \frac{3}{4\pi}\Bigg[-\pi\mathrm{i}\sum_{n=1}^{N}\frac{[\sin(a_0\theta_n) - a_0\theta_n\cos(a_0\theta_n)]\,J_1(a_0\theta_n)}{R(a_0\theta_n)^2}\Bigg]. \tag{235}$$

θ_m is imaginary, $H_1^{(1)}(a_0\theta_m)$ is real, and it follows that all the terms in f_{1p} are real and those in $\mathrm{i}f_{2p}$ are imaginary. Tables of Hankel functions of imaginary arguments are available (Jahnke & Emde 1945) and the series may be readily evaluated. The infinite series in θ_m converges slowly, and it is convenient to sum it as far as $m = (Q-1)$ and to approximate to the remainder. It is required to approximate to

$$S_Q = 2\pi\mathrm{i}\sum_{m=Q}^{\infty}\frac{[\sin(a_0\theta_m) - a_0\theta_m\cos(a_0\theta_m)]\,H_1^{(1)}(a_0\theta_m)}{R(a_0\theta_m)^2}, \tag{236}$$

$$a_0\theta_m = +\mathrm{i}\left[\frac{(2m-1)^2\pi^2}{4R^2} - a_0^2\right]^{\frac{1}{2}}. \tag{237}$$

If m is large enough, i.e. $m \geq Q$
$$a_0\theta_m \doteqdot \frac{\mathrm{i}(2m-1)\pi}{2R}, \tag{238}$$

and

$$S_Q \doteqdot 4\sum_{m=Q}^{\infty}\frac{[2R\sinh[(2m-1)\pi/2R] - (2m-1)\pi\cosh[(2m-1)\pi/2R]]\,H_1^{(1)}[\mathrm{i}(2m-1)\pi/2R]}{(2m-1)^2\pi}. \tag{239}$$

If m is large enough

$$H_1^{(1)}\left[\frac{\mathrm{i}(2m-1)\,\pi}{2R}\right] \doteqdot \frac{-2R^{\frac{1}{2}}\exp[-(2m-1)\,\pi/2R]}{\pi(2m-1)^{\frac{1}{2}}} \tag{240}$$

and

$$S_Q \doteqdot \frac{-8R^{\frac{3}{2}}}{\pi^2}\sum_{m=Q}^{\infty}\frac{1}{(2m-1)^{\frac{5}{2}}}+\frac{4R^{\frac{1}{2}}}{\pi}\sum_{m=Q}^{\infty}\frac{1}{(2m-1)^{\frac{3}{2}}}. \tag{241}$$

It may be shown that

$$\int_Q^{\infty}\frac{\mathrm{d}m}{(2m-1)^{\frac{3}{2}}}<\sum_{m=Q}^{\infty}\frac{1}{(2m-1)^{\frac{3}{2}}}<\int_Q^{\infty}\frac{\mathrm{d}m}{(2m-3)^{\frac{3}{2}}}, \tag{242}$$

$$\sum_{m=Q}^{\infty}\frac{1}{(2m-1)^{\frac{3}{2}}}\doteqdot\frac{1}{2}\left[\frac{1}{(2Q-1)^{\frac{1}{2}}}+\frac{1}{(2Q-3)^{\frac{1}{2}}}\right], \tag{243}$$

$$\sum_{m=Q}^{\infty}\frac{1}{(2m-1)^{\frac{5}{2}}}\doteqdot\frac{1}{6}\left[\frac{1}{(2Q-1)^{\frac{3}{2}}}+\frac{1}{(2Q-3)^{\frac{3}{2}}}\right]. \tag{244}$$

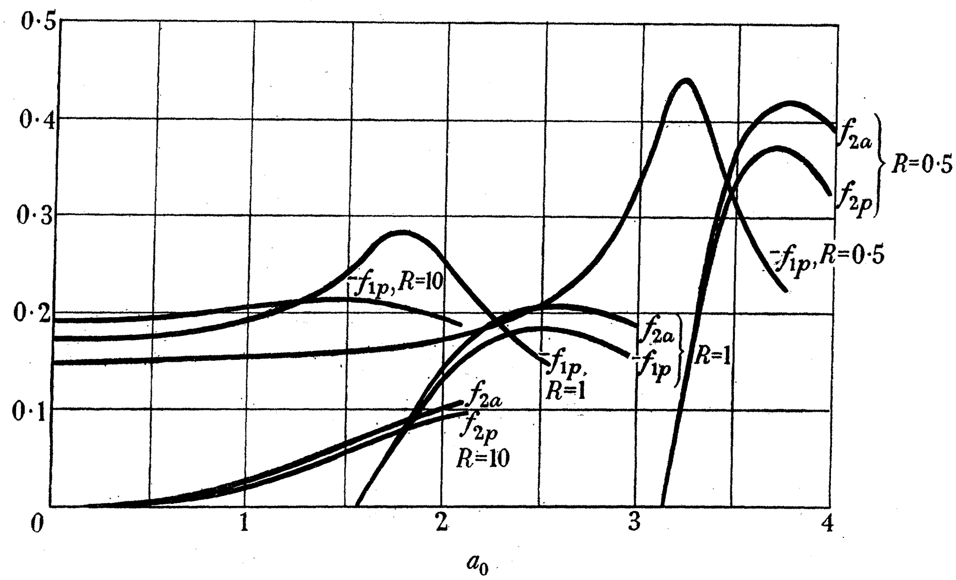

FIGURE 13. Elastic stratum, rotation in horizontal plane.

Q was chosen so that the approximation represents S_Q to within 5 % and then the overall error in f_{1p} is much less than 5 %.

The evaluated curves for different a_0 and R are shown in figure 13. The interesting fact that is shown by these curves is that resonance does not exist. Neither does it exist in the case of rotation about a horizontal axis. Instead, a maximum in f_1 occurs and it occurs at a frequency close to the first resonant frequency of a rod of the medium equal in length to the depth of the stratum, fixed at the bottom, free at the top, and oscillating in torsion. As R becomes smaller the frequency at the maximum point moves closer to this frequency and the amplitude of the maximum becomes greater. The limit is obvious and is to be expected. θ_n is zero until a certain frequency given by $a_0 \geqslant \pi/2R$ is attained. There are no free waves below this value and no energy propagation. It can be easily shown that this critical value is the first resonant frequency of the rod just mentioned.

It is probable that further maxima occur at frequencies close to $a_0 = (2m-1)\,\pi/2R$. When the stratum becomes deep this case rapidly approaches that of the semi-infinite case.

$R = 10$, as can be seen from the curves, is very close to the semi-infinite characteristic, and it is interesting to note that the sum of the free waves in the stratum has now become the imaginary part of the integral of the semi-infinite case.

Because of the great similarity in the two cases of rotation it is to be expected that rotation of the plate about a horizonal axis on a stratum has similar characteristics. The experimental investigation confirmed this.

6. Horizontal translation

Of the four modes this is the most complicated. Shear is obviously the main effect, but dilatation also exists and the results vary a little with Poisson's ratio. In order to effect a solution by the methods of this paper the boundary conditions must be simplified. It will be shown that horizontal forces over $r \leqslant r_0$ cause not only horizontal translation but also rotation of the surface. A rigid plate, subject to a horizontal force, translates horizontally but also rotates in a vertical plane. It is to be expected, and is observed experimentally, that this rotation is small. The sum of the three solutions, equations (8) to (16), is used in the solution to this case.

Semi-infinite elastic space

(*a*) *Dynamic case*

$$u_1+u_3 = \left[\frac{-A(x)\,e^{-\alpha z}}{h^2}+\frac{C(x)\,e^{-\beta z}}{k^2}\right]\frac{\partial J_1(xr)}{\partial r}\,\frac{\cos}{\sin}\,\theta\,e^{ipt}, \tag{245}$$

$$v_1+v_3 = \left[\frac{A(x)\,e^{-\alpha z}}{h^2}-\frac{C(x)\,e^{-\beta z}}{k^2}\right]\frac{J_1(xr)}{r}\,\frac{\sin}{-\cos}\,\theta\,e^{ipt}, \tag{246}$$

$$w_1+w_3 = \left[\frac{A(x)\,\alpha\,e^{-\alpha z}}{h^2}-\frac{C(x)\,x^2\,e^{-\beta z}}{\beta k^2}\right]J_1(xr)\,\frac{\cos}{\sin}\,\theta\,e^{ipt}, \tag{247}$$

$$\widehat{zz}_{1+3} = \mu\left[A(x)\left(\frac{\lambda}{\mu}-\frac{2\alpha^2}{h^2}\right)e^{-\alpha z}+\frac{C(x)\,2x^2\,e^{-\beta z}}{k^2}\right]J_1(xr)\,\frac{\cos}{\sin}\,\theta\,e^{ipt}, \tag{248}$$

$$\widehat{rz}_{1+3} = \mu\left[\frac{A(x)\,2\alpha\,e^{-\alpha z}}{h^2}-\frac{C(x)\,(\beta^2+x^2)\,e^{-\beta z}}{\beta k^2}\right]\frac{\partial J_1(xr)}{\partial r}\,\frac{\cos}{\sin}\,\theta\,e^{ipt}, \tag{249}$$

$$\widehat{z\theta}_{1+3} = \mu\left[\frac{-A(x)\,2\alpha\,e^{-\alpha z}}{h^2}+\frac{C(x)\,(\beta^2+x^2)\,e^{-\beta z}}{\beta k^2}\right]\frac{J_1(xr)}{r}\,\frac{\sin}{-\cos}\,\theta\,e^{ipt}. \tag{250}$$

For convenience these solutions are written as

$$u_{1+3} = F_1(x,z)\,\frac{\partial J_1(xr)}{\partial r}\,\frac{\cos}{\sin}\,\theta\,e^{ipt}, \tag{251}$$

$$v_{1+3} = -F_1(x,z)\,\frac{J_1(xr)}{r}\,\frac{\sin}{-\cos}\,\theta\,e^{ipt}, \tag{252}$$

$$w_{1+3} = F_2(x,z)\,J_1(xr)\,\frac{\cos}{\sin}\,\theta\,e^{ipt}, \tag{253}$$

$$\widehat{zz}_{1+3} = F_3(x,z)\,J_1(xr)\,\frac{\cos}{\sin}\,\theta\,e^{ipt}, \tag{254}$$

$$\widehat{rz}_{1+3} = -F_4(x,z)\,\frac{\partial J_1(xr)}{\partial r}\,\frac{\cos}{\sin}\,\theta\,e^{ipt}, \tag{255}$$

$$\widehat{z\theta}_{1+3} = F_4(x,z)\,\frac{J_1(xr)}{r}\,\frac{\sin}{-\cos}\,\theta\,e^{ipt}. \tag{256}$$

Add on the second solution, i.e.

$$\left.\begin{aligned} u_2 &= \frac{B(x)}{x^2}\frac{J_1(xr)}{r}\,\mathrm{e}^{-\beta z}\,{\cos \atop \sin}\,\theta\,\mathrm{e}^{\mathrm{i}pt}, \\ v_2 &= \frac{-B(x)}{x^2}\frac{\partial J_1(xr)}{\partial r}\,\mathrm{e}^{-\beta z}\,{\sin \atop -\cos}\,\theta\,\mathrm{e}^{\mathrm{i}pt}, \\ w_2 &= 0, \end{aligned}\right\} \tag{257}$$

and the solution on the free surface $z = 0$ becomes

$$u(r, 0) = \left[F_1(x, 0)\frac{\partial J_1(xr)}{\partial r} + \frac{B(x)}{x^2}\frac{J_1(xr)}{r}\right]{\cos \atop \sin}\,\theta\,\mathrm{e}^{\mathrm{i}pt}, \tag{258}$$

$$v(r, 0) = \left[-F_1(x, 0)\frac{J_1(xr)}{r} - \frac{B(x)}{x^2}\frac{\partial J_1(xr)}{\partial r}\right]{\sin \atop -\cos}\,\theta\,\mathrm{e}^{\mathrm{i}pt}, \tag{259}$$

$$w(r, 0) = F_2(x, 0)\,J_1(xr)\,{\cos \atop \sin}\,\theta\,\mathrm{e}^{\mathrm{i}pt}, \tag{260}$$

$$\widehat{zz}(r, 0) = F_3(x, 0)\,J_1(xr)\,{\cos \atop \sin}\,\theta\,\mathrm{e}^{\mathrm{i}pt}, \tag{261}$$

$$\widehat{rz}(r, 0) = \left[-F_4(x, 0)\frac{\partial J_1(xr)}{\partial r} - \frac{\mu\beta B(x)\,J_1(xr)}{x^2 r}\right]{\cos \atop \sin}\,\theta\,\mathrm{e}^{\mathrm{i}pt}, \tag{262}$$

$$\widehat{z\theta}(r, 0) = \left[F_4(x)\frac{J_1(xr)}{r} + \frac{\mu\beta B(x)}{x^2}\frac{\partial J_1(xr)}{\partial r}\right]{\sin \atop -\cos}\,\theta\,\mathrm{e}^{\mathrm{i}pt}. \tag{263}$$

It is possible to equate $\widehat{zz} = 0$ which expresses $C(x)$ in terms of $A(x)$ and then to choose $A(x)$ and $B(x)$ so that the shear stress over $r \leqslant r_0$ is some particular distribution. If this is done then it is easily seen that $w(r, 0)$ is finite and changes sign with r, i.e. a rotation of the surface takes place. The expressions for the displacements then involve the characteristic equation $f(x)$. A great simplification results if, instead of equating $\widehat{zz} = 0$, we equate $w(r, 0) = 0$ everywhere. $\widehat{zz}$ is now finite, meaning that vertical forces have been added to keep the surface flat. It is observed experimentally that the vertical displacements caused by horizontal forces are small, and the effect on the horizontal displacements of preventing these vertical displacements will be this degree of smallness squared. Put $w = 0$, i.e.

$$\left.\begin{aligned} F_2(x, 0) &= 0, \\ C(x) &= \frac{A(x)\,\alpha\beta k^2}{x^2 h^2} \end{aligned}\right\} \tag{264}$$

and choose
$$F_4(x) = \frac{-2\mu\alpha A(x)}{h^2} + \frac{\mu C(x)\,(x^2+\beta^2)}{\beta k^2} = \frac{\mu\beta B(x)}{x^2}, \tag{265}$$

i.e.
$$B(x) = \frac{-A(x)\,x^2 k^2 \alpha}{\beta x^2 h^2}.$$

It can be shown from the recurrence relations of Bessel functions that

$$\frac{\partial J_1(xr)}{\partial r} + \frac{J_1(xr)}{r} = xJ_0(xr). \tag{266}$$

The shear stresses follow as

$$\widehat{rz} = -F_4(x)\, xJ_0(xr) \begin{matrix}\cos\\ \sin\end{matrix}\, \theta\, \mathrm{e}^{\mathrm{i}pt}, \tag{267}$$

$$\widehat{z\theta} = F_4(x)\, xJ_0(xr) \begin{matrix}\sin\\ -\cos\end{matrix}\, \theta\, \mathrm{e}^{\mathrm{i}pt}. \tag{268}$$

There is a further difficulty regarding the shear stress distribution to assume. A consideration of the static solutions formed by taking the limit of equations (258) to (263) as $p \to 0$ will show that the previous methods will not apply in this case without further simplifying assumptions. In equations (258), (259), (262) and (263), put

$$F_1(x, 0) = \frac{B(x)}{x^2}, \quad F_4(x, 0) = \frac{\mu\beta B(x)}{x^2}. \tag{269}$$

In the static case it follows that the displacements and stresses are now of the form

$$u = F_1(x, 0)\, xJ_0(xr) \begin{matrix}\cos\\ \sin\end{matrix}\, \theta, \tag{270}$$

$$v = -F_1(x, 0)\, xJ_0(xr) \begin{matrix}\sin\\ -\cos\end{matrix}\, \theta, \tag{271}$$

$$\widehat{rz} = -F_1(x, 0)\, x^2J_0(xr) \begin{matrix}\cos\\ \sin\end{matrix}\, \theta, \tag{272}$$

$$\widehat{z\theta} = F_1(x, 0)\, x^2J_0(xr) \begin{matrix}\sin\\ -\cos\end{matrix}\, \theta. \tag{273}$$

The boundary conditions to be satisfied are that

$$\left.\begin{aligned} u &= d\sin\theta \quad (r \leqslant r_0),\\ v &= d\cos\theta \quad (r \leqslant r_0),\\ \widehat{rz} &= \widehat{z\theta} = 0 \quad (r > r_0),\end{aligned}\right\} \tag{274}$$

where d is the horizontal displacement of the rigid plate. Generalize equations (270) to (273) by integrating with respect to x, from 0 to ∞, and the boundary conditions are satisfied if $F_1(x, 0)$ can be chosen so as to satisfy the dual integral equations

$$\left.\begin{aligned} \int_0^\infty F_1(x, 0)\, xJ_0(xr)\, \mathrm{d}x &= d \quad (r \leqslant r_0),\\ \int_0^\infty F_1(x, 0)\, x^2J_0(xr)\, \mathrm{d}x &= 0 \quad (r > r_0).\end{aligned}\right\} \tag{275}$$

These are the same two dual equations as occur in statical vertical translation and they yield a stress distribution given by,

$$\widehat{rz} = \frac{K\sin\theta}{(r_0^2 - r^2)^{\frac{1}{2}}}, \quad \widehat{z\theta} = \frac{K\cos\theta}{(r_0^2 - r^2)^{\frac{1}{2}}} \quad (r \leqslant r_0). \tag{276}$$

This solution makes both $\widehat{zz}$ and w finite and does not fulfil the condition $w = 0$ as already assumed. However, we will take it as being a reasonable approximation to the static stress under the previous conditions assumed.

Return to the dynamic equations (267) and (268), and when $r \leqslant r_0$,

$$\widehat{rz} = \int_0^\infty F_4(x)\, xJ_0(xr)\, \mathrm{d}x \sin\theta\, \mathrm{e}^{\mathrm{i}pt} = \frac{K\sin\theta\, \mathrm{e}^{\mathrm{i}pt}}{(r_0^2 - r^2)^{\frac{1}{2}}}, \tag{277}$$

$$\widehat{z\theta} = \int_0^\infty F_4(x)\, xJ_0(xr)\, \mathrm{d}x \cos\theta\, \mathrm{e}^{\mathrm{i}pt} = \frac{K\cos\theta\, \mathrm{e}^{\mathrm{i}pt}}{(r_0^2 - r^2)^{\frac{1}{2}}}. \tag{278}$$

By the Fourier–Bessel theorem,

$$\frac{K}{(r_0^2 - r^2)^{\frac{1}{2}}} \quad (r \leqslant r_0), \quad = \int_0^\infty \sin(xr_0)\, J_0(xr)\, \mathrm{d}x. \tag{279}$$

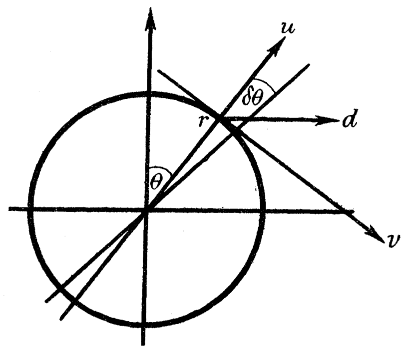

FIGURE 14

Comparison of coefficients yields

$$\frac{B(x)}{x^2} = \frac{K\sin(xr_0)}{\mu\beta x}, \tag{280}$$

$$\frac{A(x)}{h^2} = \frac{-Kx\sin(xr_0)}{\mu k^2\alpha}, \tag{281}$$

$$\frac{C(x)}{k^2} = \frac{-K\beta\sin(xr_0)}{\mu k^2\alpha}. \tag{282}$$

If P is the horizontal force on the plate, then $K = P/2\pi r_0$ and the displacements become

$$u = \frac{P\,\mathrm{e}^{\mathrm{i}pt}\sin\theta}{2\pi\mu r_0}\int_0^\infty \left[\frac{x\sin(xr_0)}{k^2\alpha} - \frac{\beta\sin(xr_0)}{k^2x}\right]\frac{\partial J_1(xr)}{\partial r} + \frac{\sin(xr_0)\, J_1(xr)\,\mathrm{d}x}{\beta xr}, \tag{283}$$

$$v = \frac{P\,\mathrm{e}^{\mathrm{i}pt}\cos\theta}{2\pi\mu r_0}\int_0^\infty \left[\frac{x\sin(xr_0)}{k^2\alpha} - \frac{\beta\sin(xr_0)}{k^2x}\right]\frac{J_1(xr)}{r} + \frac{\sin(xr_0)\,\partial J_1(xr)\,\mathrm{d}x}{\beta x\partial r}. \tag{284}$$

'Average' these displacements in the direction $\theta = \frac{1}{2}\pi$ in the same manner as before, i.e. integrate the displacement of a point multiplied by the stress acting at the point over the area $r \leqslant r_0$. If U_a is the 'average displacement' of the plate in the direction of the horizontal force, i.e. $\theta = \frac{1}{2}\pi$, then

$$U_aP = \int_0^{2\pi}\mathrm{d}\theta\int_0^{r_0}\mathrm{d}r\left[\frac{Pr(u\sin\theta + v\cos\theta)}{2\pi r_0(r_0^2 - r^2)^{\frac{1}{2}}}\right]. \tag{285}$$

The order of integration may be interchanged and finally

$$\begin{aligned} U_a &= \frac{P\,\mathrm{e}^{\mathrm{i}pt}}{4\mu\pi r_0^2}\int_0^\infty\left[\frac{x^2 - \alpha\beta}{k^2\alpha x} + \frac{1}{\beta x}\right]\sin^2(xr_0)\,\mathrm{d}x \\ &= \frac{P\,\mathrm{e}^{\mathrm{i}pt}}{4\mu\pi r_0}\int_0^\infty\left[\frac{\theta^2 - (\theta^2 - \tau^2)^{\frac{1}{2}}(\theta^2 - 1)^{\frac{1}{2}}}{(\theta^2 - \tau^2)^{\frac{1}{2}}} + \frac{1}{(\theta^2 - 1)^{\frac{1}{2}}}\right]\frac{\sin^2(a_0\theta)\,\mathrm{d}\theta}{a_0\theta}. \end{aligned} \tag{286}$$

There are no free waves to be added. Equate all surface stresses to zero and all the displacements vanish.

Evaluation follows as in the previous work. Consider

$$J=\int_{-\infty}^{\infty}\left[\frac{\theta^2-(\theta^2-\tau^2)^{\frac{1}{2}}(\theta^2-1)^{\frac{1}{2}}}{(\theta^2-\tau^2)^{\frac{1}{2}}}+\frac{1}{(\theta^2-1)^{\frac{1}{2}}}\right]\frac{e^{ia_0\theta}\sin(a_0\theta)\,d\theta}{a_0\theta}. \tag{287}$$

Integrate J around the infinite semicircle contour, changing the signs of the radicals at the negative branch points, and

$$\begin{aligned}I&=\int_0^{\infty}\left[\frac{\theta^2-(\theta^2-\tau^2)^{\frac{1}{2}}(\theta^2-1)^{\frac{1}{2}}}{(\theta^2-\tau^2)^{\frac{1}{2}}}+\frac{1}{(\theta^2-1)^{\frac{1}{2}}}\right]\frac{\sin^2(a_0\theta)\,d\theta}{a_0\theta}\\&=\left[\int_0^{\tau}\frac{\theta^2\sin(2a_0\theta)\,d\theta}{(\tau^2-\theta^2)^{\frac{1}{2}}2a_0\theta}+\int_0^1(1-\theta^2)^{\frac{1}{2}}\frac{\sin(2a_0\theta)\,d\theta}{2a_0\theta}+\int_0^1\frac{\sin(2a_0\theta)\,d\theta}{(1-\theta^2)^{\frac{1}{2}}2a_0\theta}\right]\\&\quad-i\left[\int_0^{\tau}\frac{\theta^2\sin^2(a_0\theta)\,d\theta}{(\tau^2-\theta^2)^{\frac{1}{2}}a_0\theta}+\int_0^1\frac{(1-\theta^2)^{\frac{1}{2}}\sin^2(a_0\theta)\,d\theta}{a_0\theta}+\int_0^1\frac{\sin^2(a_0\theta)\,d\theta}{(1-\theta^2)^{\frac{1}{2}}a_0\theta}\right].\end{aligned} \tag{288}$$

These are integrated by the substitutions $\theta=\tau\sin\psi$, $\theta=\sin\psi'$.

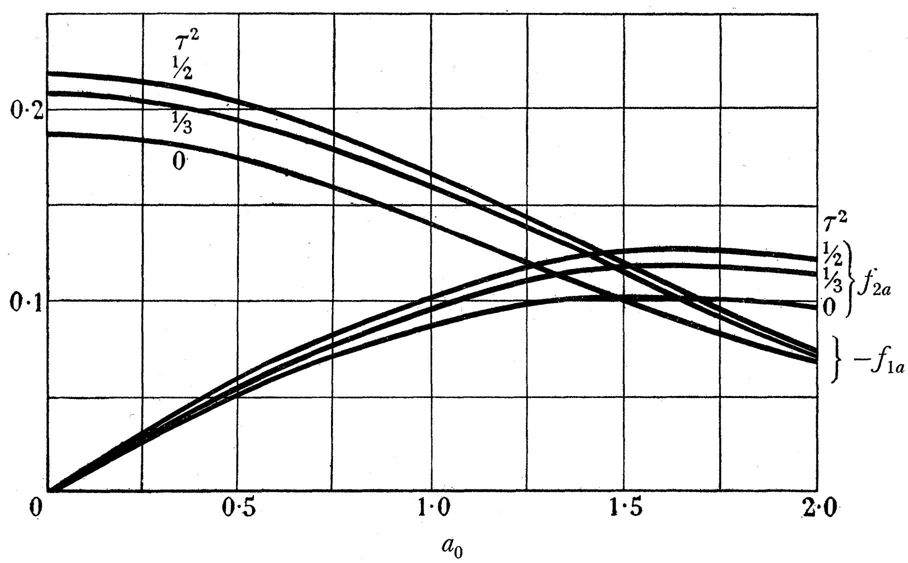

FIGURE 15. Semi-infinite space, horizontal translation.

Finally,

$$\begin{aligned}I=&\begin{bmatrix}\frac{1}{4}\pi(3+\tau^2)-(0{\cdot}392\tau^4+0{\cdot}655)\,a_0^2+(0{\cdot}0655\tau^6+0{\cdot}0913)\,a_0^4\\-(0{\cdot}00550\tau^8+0{\cdot}00701)\,a_0^6+(0{\cdot}00027\tau^{10}+0{\cdot}00033)\,a_0^8\\-(0{\cdot}00001\tau^{12}+0{\cdot}00001)\,a_0^{10}+\dots\end{bmatrix}\\&-i\begin{bmatrix}(0{\cdot}667\tau^3+1{\cdot}333)\,a_0-(0{\cdot}177\tau^5+0{\cdot}266)\,a_0^3\\+(0{\cdot}0203\tau^7+0{\cdot}0271)\,a_0^5-(0{\cdot}0013\tau^9+0{\cdot}0016)\,a_0^7\\+(0{\cdot}000052\tau^{11}+0{\cdot}00006)\,a_0^9+\dots\end{bmatrix}.\end{aligned} \tag{289}$$

$$\left.\begin{aligned}U_a&=\frac{P\,e^{ipt}}{\mu r_0}(f_{1a}+if_{2a}),\\f_{1a}+if_{2a}&=I/4\pi.\end{aligned}\right\} \tag{290}$$

τ^2 varies from 0 to $\frac{1}{2}$ for the full range of Poisson's ratio and the series shows that I does not vary as much with τ^2 as it does in vertical translation. This is to be expected as the forces here are predominantly shearing forces. Figure 15 shows that the characteristics are similar to those of vertical translation.

(*b*) *Static displacement*

Put $a_0 = 0$ and

$$U_a = \frac{P(3+\tau^2)}{16\mu r_0}. \tag{291}$$

This relation shows that the horizontal static stiffness of the plate does not vary very much with Poisson's ratio.

By the method used previously it may be shown that horizontal displacements caused by the shear stress distribution of the semi-infinite case contain terms such as

$$T = \int_0^\infty \frac{\sin(xr_0)\sinh(\beta\delta)\,J_1(xr)\,dx}{x\beta\cosh(\beta\delta)\,r}, \tag{292}$$

$$\begin{aligned}\coth[\delta(x^2-k^2)^{\frac{1}{2}}] &= -\mathrm{i}\cot[\delta(k^2-x^2)^{\frac{1}{2}}]\\ &= -\mathrm{i}\cot\left[\delta k\left(1-\frac{x^2}{2k^2}+-\right)\right] \quad \text{for small } x.\end{aligned} \tag{293}$$

Put

$$\delta = \frac{(2n-1)\,\pi}{2k}.$$

Expand to the first order in x^2, and

$$\mathrm{i}\cot\left[\delta k\left(1-\frac{x^2}{2k^2}+\dots\right)\right] \doteqdot \frac{-\mathrm{i}\,(2n-1)\,\pi x^2}{4k^2},$$

$$T = \int_0^\epsilon \frac{xr_0\,4k^2xr\,dx}{-\mathrm{i}(2n-1)\,\pi x^2kx\,2r} + \int_\epsilon^\infty (\quad)\,dx, \tag{294}$$

where ϵ is small. The first integral, having an integrand of order x^{-1}, diverges, i.e. the plate resonates, when

$$\left.\begin{aligned}\delta &= \frac{(2n-1)\,\pi}{2k} = \frac{(2n-1)\,\pi\mu^{\frac{1}{2}}}{2p\rho^{\frac{1}{2}}},\\ p &= \frac{(2n-1)\,\pi}{2\delta}\left(\frac{\mu}{\rho}\right)^{\frac{1}{2}}.\end{aligned}\right\} \tag{295}$$

These frequencies are those of a block fixed at the bottom, free at the top and oscillating in shear.

7. Damping

It has been seen that propagation of energy to infinity provides a damping of the motion of the plate. If the medium possesses true damping the energy is dissipated in the medium.

It may be shown that a form of damping in the medium may be represented by the addition of a small imaginary term to the modulus of rigidity, μ. There are now no real roots of $f(x) = 0$, i.e. no free waves and the integral expressions for the displacements have unique values. A contour integration of these integrals shows that the same values of f_1 and f_2 arise together with small additional terms representing the effect of the damping.

The free-wave term which is the major part of f_2 now appears automatically. If the damping is small the additional terms are small and the values of f_1 and f_2 of the undamped case are substantially those of the damped case. A form of viscous damping may be represented by replacing μ by $\mu(1+\mathrm{i}a_0K_0')$ and a form of Coulomb damping by $\mu(1+\mathrm{i}K_0)$.

It may be shown that the changes in f_1 and f_2 due to damping are given by:

(*a*) *Viscous damping*

$$\left.\begin{aligned}\Delta' f_1 &= a_0K_0'\left[f_2+\frac{a_0}{2}\frac{\partial f_2}{\partial a_0}+\frac{\partial f_2}{\partial \nu}\nu(1-2\nu)\right],\\ \Delta' f_2 &= a_0K_0'\left[-f_1-\frac{a_0}{2}\frac{\partial f_1}{\partial a_0}-\frac{\partial f_1}{\partial \nu}\nu(1-2\nu)\right].\end{aligned}\right\} \tag{296}$$

(*b*) *Coulomb damping*

Replace a_0K_0' by K_0. These deviations are shown in figure 16. It is to be noticed that f_1 becomes numerically smaller and f_2 greater for small values of a_0.

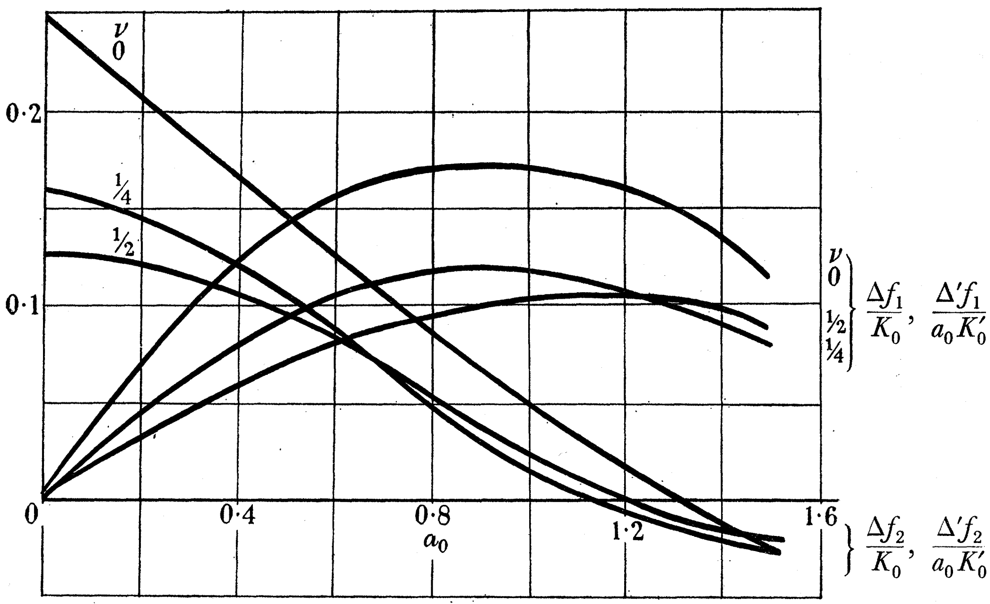

FIGURE 16. Damping curves, semi-infinite space, vertical translation.

8. Note on experimental work

A comparison of these theoretical results with experimental ones gave a very close agreement. The experimental work is being published separately, but a short note on the technique is not out of place here.

Sheets of foam rubber, 1 in. in thickness and glued together to form a block 3 ft. square and 14 in. deep, proved to be a satisfactory 'elastic half-space' for a base radius of 1 cm and the range of frequency implied by the theoretical curves. Repeated reflexions of the transmitted waves from the square boundary were sufficiently attenuated by the natural damping of the material as to be insignificant by the time they reached the centre again. The block thus behaved as though it were unbounded.

A small fibre disk on the centre of the upper face of the rubber was excited by a coil attached to it carrying a variable frequency current and oscillating in a constant field.

As foam rubber is of low density and the radius of the disk necessarily small, the mass associated with the disk also had to be small to provide relevant 'b' values. Consequently a means of measuring the dynamic displacements of the disk without introducing further inertia was needed. A capacity pick-up giving oscilloscope deflexions proportional to disk displacements achieved this requirement satisfactorily.

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